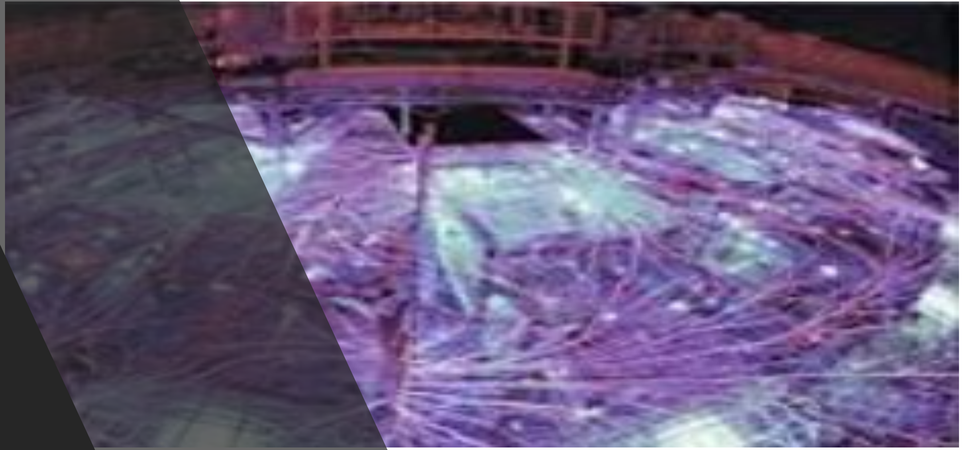
PENY. PERSAMAAN DIFFERENSIAL

DENGAN OPERATOR D

MATEMATIKA REKAYASA 1

AULIA SITI AISJAH – TEKNIK FISIKA ITS



Operator D

$$Dy = y' = \frac{dy}{dx}.$$

$$L(cy + kw) = cLy + kLw.$$

Contoh

$$D^2y = D(Dy) = y''$$

$$y'' + ay' + by = 0$$

$$Iy = y.$$

$$Le^{\lambda}(x) = P(D)e^{\lambda}(x) = (D^2 + aD + bI)e^{\lambda}(x)$$

$$= (\lambda^2 + a\lambda + b)e^{\lambda x} = P(\lambda)e^{\lambda x} = 0.$$

$$L = P(D) = D^2 + aD + bI,$$

$$Ly = P(D)y = (D^2 + aD + bI)y = 0.$$

L operator linier

Factorization, Solution of an ODE

Factor $P(D) = D^2 - 3D - 40I$ and solve $P(D)y = 0$.

Solution. $D^2 - 3D - 40I = (D - 8I)(D + 5I)$ because $I^2 = I$. Now $(D - 8I)y = y' - 8y = 0$ has the solution $y_1 = e^{8x}$. Similarly, the solution of $(D + 5I)y = 0$ is $y_2 = e^{-5x}$. This is a basis of $P(D)y = 0$ on any interval. From the factorization we obtain the ODE, as expected,

$$(D - 8I)(D + 5I)y = (D - 8I)(y' + 5y) = D(y' + 5y) - 8(y' + 5y)$$

$$= y'' + 5y' - 8y' - 40y = y'' - 3' - 40y = 0.$$

Verify that this agrees with the result of our method in Sec. 2.2. This is not unexpected because we factored $P(D)$ in the same way as the characteristic polynomial $P(\lambda) = \lambda^2 - 3\lambda - 40$. ■

$$\frac{d^2y}{dx^2} - 2\,\frac{dy}{dx} + y = 0$$

$$(D^2 - 2D + 1)\,y = 0$$

$$Or \quad (D-1)^2\,y = 0$$

$$Let \quad (D-1)\,y = u$$

$$Then \quad (D-1)\,u = 0$$

$$\therefore \quad u = A\,e^x$$

$$\therefore \quad (D-1)\,y = A\,e^x$$

$$\frac{dy}{dx} - y = A\,e^x$$

$$y\,e^{-x} = Ax + B$$

$$\therefore \quad \mathbf{y = (Ax + B)\,e^x}$$

Sifat operator D

$$(D-a) = e^{ax} D e^{-ax}$$

$$(D-a)^n = e^{ax} D^n e^{-ax}$$

$$L(y) = (aD^2 + bD + c)y = \phi(D)y = 0,$$

$$\phi(D) = (aD^2 + bD + c)$$

$$\frac{dy}{dx} - ay = e^{ax}\left(\frac{d}{dx}(e^{-ax}y)\right)$$

2.2 Cases (I) ($b^2 - 4ac > 0$)

$$\phi(D) = (D-r_1)(D-r_2)$$

$$L(y) = (D-r_1)(D-r_2)y = 0.$$

$$z = (D-r_2)y,$$

$$L(y) = (D - r_1)(D - r_2)y = 0.$$

$$z = (D - r_2)y,$$

$$(D - r_1)z = \mathrm{e}^{r_1} D \mathrm{e}^{-r_1} z = 0,$$

$$\mathrm{e}^{r_2} z = A, \quad z = A\mathrm{e}^{r_1}.$$

$$(D - r_2)y = \mathrm{e}^{r_2} D \mathrm{e}^{-r_2} y = z = A\mathrm{e}^{r_1},$$

$$D\left(\mathrm{e}^{-r_2} y\right) = z = A\mathrm{e}^{r_1 - r_2}$$

$$y = \tilde{A}\mathrm{e}^{r_1} + B\mathrm{e}^{r_2},$$

$$\tilde{A} = \frac{A}{(r_1 - r_2)},$$

$$L(y) = (D - r_1)(D - r_2) \cdots (D - r_n)y = 0,$$

$$r_i \neq r_j, (i \neq j)$$

$$L(y) = (D - r_1)(D - r_2) \cdots (D - r_n)y = 0,$$

$$(D - r_i)y_i = 0, \qquad (i = 1, 2, \cdots, n)$$

$$y(x) = y_1(x) + y_2(x) + \cdots + y_n(x).$$

2.3 Cases (II) ($b^2 - 4ac = 0$)

$r_1 = r_2.$

$\phi(D) = (D - r_1)^2$

$L(y) = (D - r_1)^2 y = 0.$

$$(D - r_1)^2 y = e^{r_1 x} D^2 e^{-r_1 x} y =$$

$$D\left(e^{-r_2 x} y\right) = z = A e^{(r_1 - r_2)x}$$

$$D^2\left(e^{-r_1 x} y\right) = 0.$$

$$\left(e^{-r_1 x} y\right) = A + Bx,$$

$$y = (A + Bx)e^{r_1 x}.$$

$$L(y) = (D - r_1)^n y = 0.$$

$$y = \left(A_1 + A_2 x + \cdots + A_n x^{n-1}\right)e^{r_1 x}.$$

2.4 Cases (III) ($b^2 - 4ac < 0$)

$r_{1,2} = \lambda \pm \mathrm{i}\mu.$

$$\phi(D) = (D-\lambda)^2 + \mu^2,$$

$$L(y) = ((D-\lambda)^2 + \mu^2)y = 0.$$

$$L(z) = (D^2 + \mu^2)z = 0.$$

$$D(\cos\mu x) = -\mu \sin x, \qquad D(\sin x) = \mu \cos x,$$

$$z(x) = A\cos\mu x + B\sin\mu x.$$

$$(\mathrm{e}^{\lambda x} D^2 \mathrm{e}^{-\lambda x} + \mu^2)y = 0.$$

$$D^2(\mathrm{e}^{-\lambda x}y) + \mu^2 \mathrm{e}^{-\lambda x} y = (D^2 + \mu^2)\mathrm{e}^{-\lambda x} y = 0.$$

$$\mathrm{e}^{-\lambda x} y(x) = A\cos\mu x + B\sin\mu x,$$

$$y(x) = \mathrm{e}^{\lambda x}\left(A\cos\mu x + B\sin\mu x\right)$$

Dengan r1 dan r2 adalah bil. kompleks

$$y(x) = \mathrm{e}^{\lambda x}\left(A\mathrm{e}^{\mathrm{i}\mu x} + B\mathrm{e}^{-\mathrm{i}\mu x}\right)$$

Example 1. $y'' + 2y' + y = x$

$$(D^2 + 2D + I)(y) = x.$$

$$(D^2 + 2D + I) = \phi(D)$$

$$(D + I)^2 = \left(e^{-x} D e^{x}\right)\left(e^{-x} D e^{x}\right) = e^{-x} D^2 e^{x}.$$

$$e^{-x} D^2 e^{x}(y) = x$$

$$\frac{d^2}{dx}(e^{x} y) = xe^{-x}$$

$$e^{x} y = xe^{-x} - 2e^{-x} + Ax + B, \quad y = x - 2 + Axe^{-x} + Be^{-x}.$$

Example 2. $y'' - 3y' + 2y = e^x.$

$$(D^2 - 3D + 2I)(y) = e^x.$$

$$(D^2 - 3D + 2I) = (D - I)(D - 2I)$$

$$(D - I)(D - 2I)(y) = e^x.$$

$$z = (D - 2I)$$

$$(D - I)(z) = e^x,$$

$$z = xe^x + Ae^x.$$

$$z = (D - 2I)(y) \qquad y' - 2y = xe^x + Ae^x$$

$$y = e^x - xe^x - Ae^x + Be^{2x}.$$

Penyelesaian Umum

Penyelesaian Partikulir

Example 3

$$y'' + 2y' + 5y = \sin(x)$$

$$(D^2 + 2D + 5I)(y) = \sin(x).$$

$$D^2 + 2D + 5I = (D + I)^2 + 4I$$

Penyelesaian Umum: y (x)

$$Ae^{-x}\cos(2x) + Be^{-x}\sin(2x).$$

Example 4.

$$y''' - 3y'' + 7y' - 5y = 0, \quad y(0) = 1, y'(0) = y''(0) = 0$$

$$(D^3 - 3D^2 + 7D - 3)(y) = 0.$$

$$\phi(r) = r^3 - 3r^2 + 7r - 5 = (r-1)(r^2 - 2r + 5) = (r-1)\left[(r-1)^2 + 4\right]$$

$$\begin{aligned} L(y) &= (D^3 - 3D^2 + 7D - 3)(y) \\ &= (D-1)\left[(D-1)^2 + 4\right](y) \\ &= \left[(D-1)^2 + 4\right](D-1)(y) \\ &= 0. \end{aligned}$$

$$(D-1)(y) = 0, \qquad \left[(D-1)^2 + 4\right](y) = 0,$$

$$y(x) = c_1 e^x, \qquad y(x) = c_2 e^x \cos(2x) + c_3 e^x \sin(2x),$$

$$y = c_1 e^x + c_2 e^x \cos(2x) + c_3 e^x \sin(2x),$$

$$y = c_1 e^x + c_2 e^x \cos(2x) + c_3 e^x \sin(2x),$$

$$y(0) = 1, y'(0) = 0, y''(0) = 0,$$

$$c_1 + c_2 = 1,$$

$$c_1 + c_2 + 2c_3 = 0,$$

$$c_1 - 3c_2 + 4c_3 = 0,$$

$$c_1 = 5/4, c_2 = -1/4, c_3 = -1/2.$$

PD orde 2

$$my'' + ky = 0.$$

$$\omega_0 = \sqrt{\frac{k}{m}}.$$

$$y(t) = A \cos \omega_0 t + B \sin \omega_0 t$$

$$y(t) = C \cos (\omega_0 t - \delta)$$

$C = \sqrt{A^2 + B^2}$ dg $\tan \delta = B/A.$

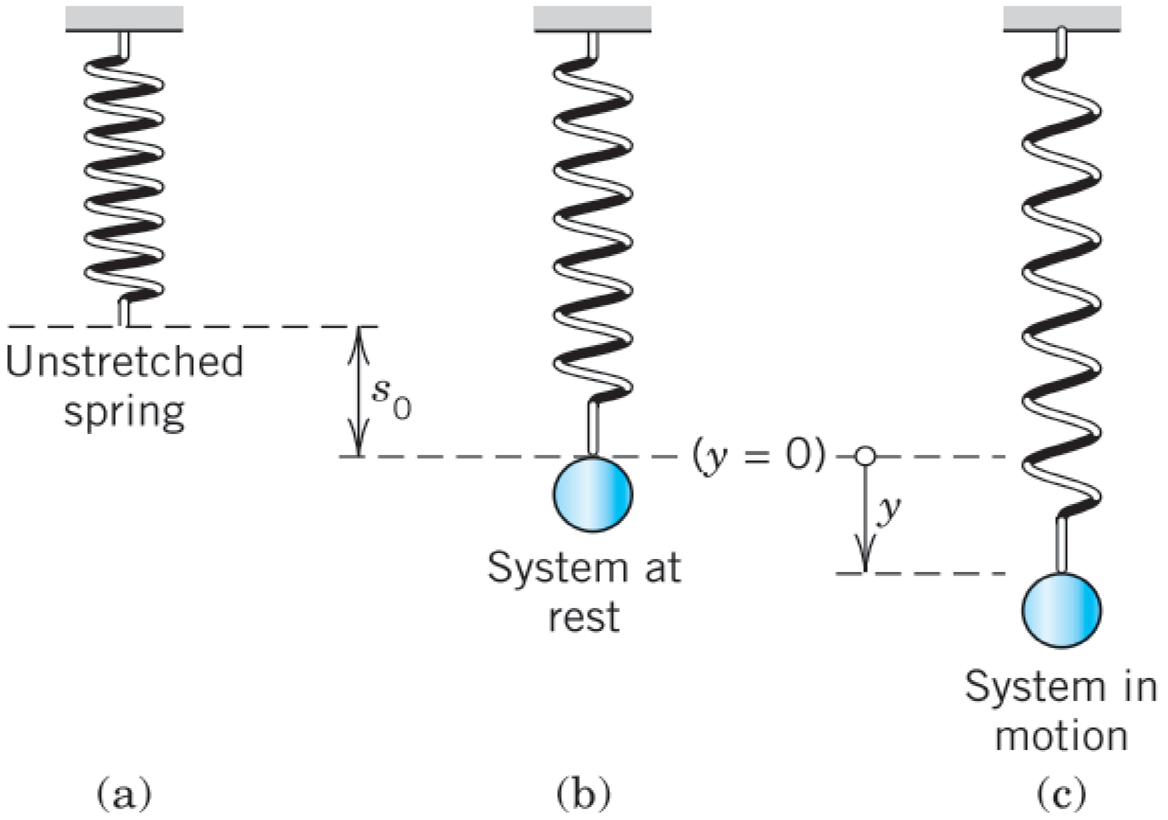


Fig. 33. Mechanical mass–spring system

$$F_1 = -ky$$

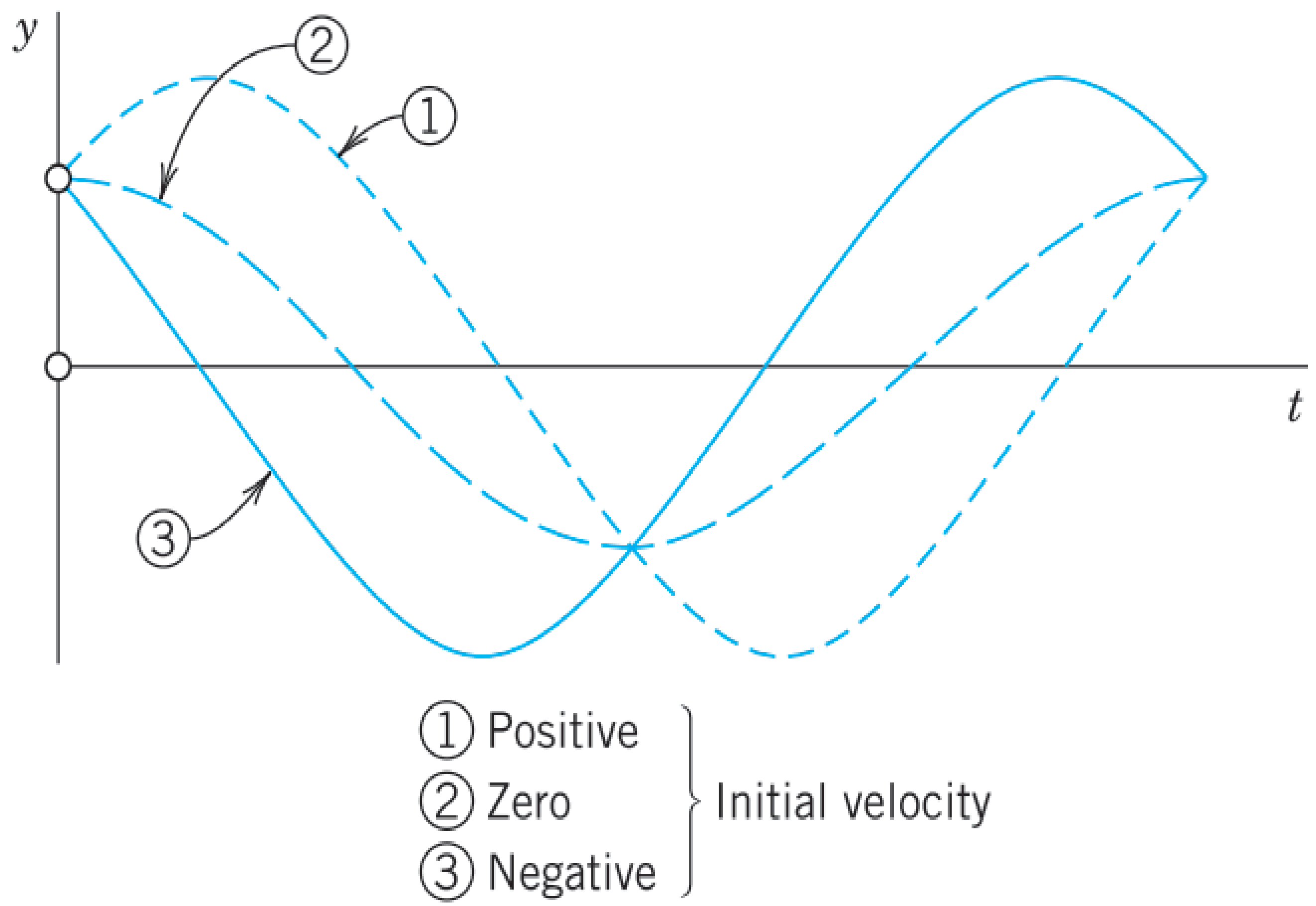


Fig. 34. Typical harmonic oscillations (4) and (4*) with the same $y(0) = A$ and different initial velocities $y'(0) = \omega_0 B$, positive ①, zero ②, negative ③

Harmonic Oscillation of an Undamped Mass–Spring System

If a mass–spring system with an iron ball of weight $W = 98$ nt (about 22 lb) can be regarded as undamped, and the spring is such that the ball stretches it 1.09 m (about 43 in.), how many cycles per minute will the system execute? What will its motion be if we pull the ball down from rest by 16 cm (about 6 in.) and let it start with zero initial velocity?

Solution. Hooke's law (1) with W as the force and 1.09 meter as the stretch gives $W = 1.09k$; thus $k = W/1.09 = 98/1.09 = 90$ [kg/sec^2] $= 90$ [nt/meter]. The mass is $m = W/g = 98/9.8 = 10$ [kg]. This gives the frequency $\omega_0/(2\pi) = \sqrt{k/m}/(2\pi) = 3/(2\pi) = 0.48$ [Hz] $= 29$ [cycles/min].

From (4) and the initial conditions, $y(0) = A = 0.16$ [meter] and $y'(0) = \omega_0 B = 0$. Hence the motion is

$$y(t) = 0.16 \cos 3t \text{ [meter]} \qquad \text{or} \qquad 0.52 \cos 3t \text{ [ft]} \qquad \text{(Fig. 35).}$$

If you have a chance of experimenting with a mass–spring system, don't miss it. You will be surprised about the good agreement between theory and experiment, usually within a fraction of one percent if you measure carefully. ■

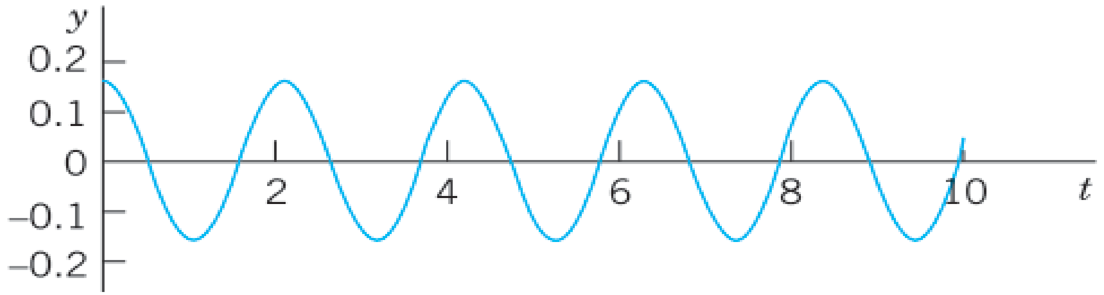


Fig. 35. Harmonic oscillation in Example 1

ODE of the Damped System

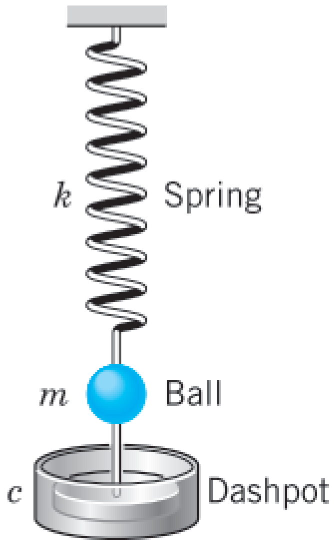


Fig. 36.
Damped system

Gaya peredam

$$F_2 = -cy',$$

Pers. Hukum Newton

$$my'' + cy' + ky = 0.$$

$$\lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0.$$

$$\lambda_1 = -\alpha + \beta, \quad \lambda_2 = -\alpha - \beta,$$

$$\alpha = \frac{c}{2m} \qquad \beta = \frac{1}{2m}\sqrt{c^2 - 4mk}.$$

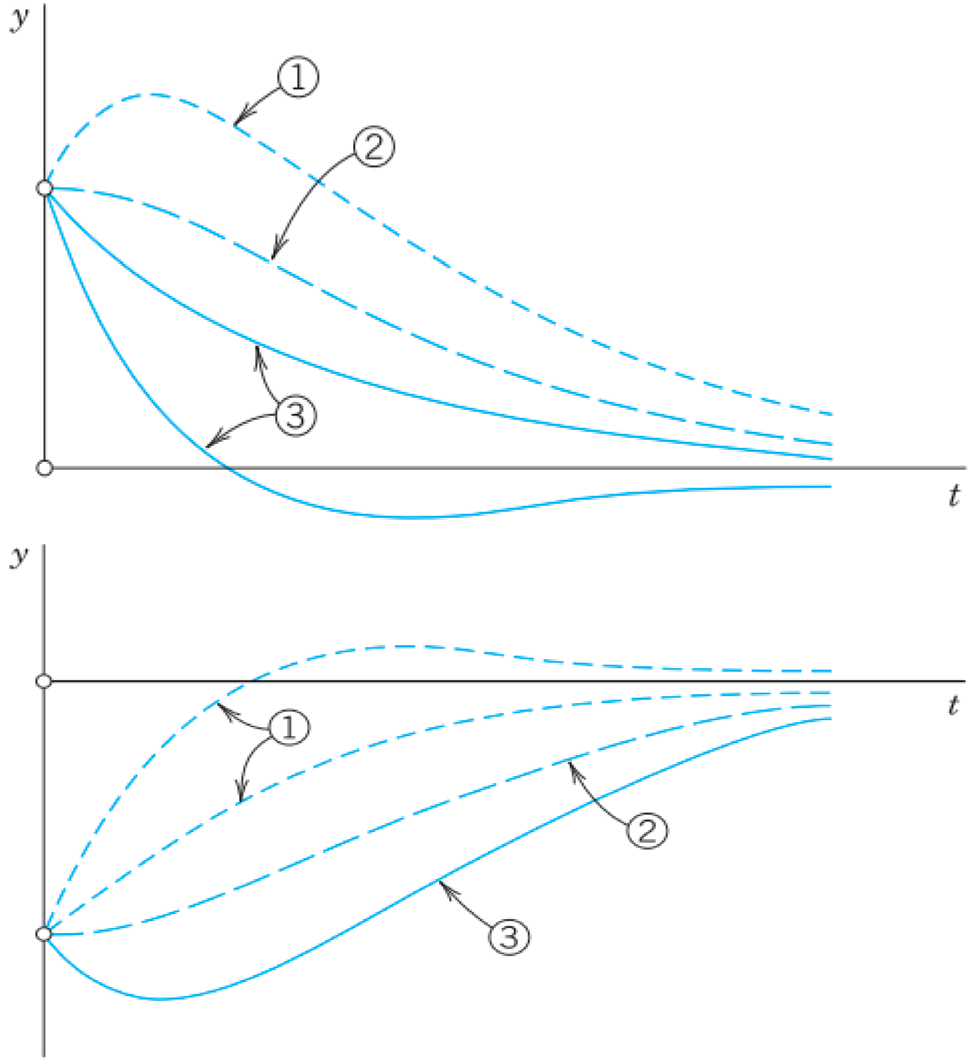
Case I.	$c^2 > 4mk$.	*Distinct real roots* λ_1, λ_2.	**(Overdamping)**
Case II.	$c^2 = 4mk$.	*A real double root.*	**(Critical damping)**
Case III.	$c^2 < 4mk$.	*Complex conjugate roots.*	**(Underdamping)**

Discussion of the Three Cases

Case I. Overdamping

$$y(t) = c_1 e^{-(\alpha-\beta)t} + c_2 e^{-(\alpha+\beta)t}.$$

① Positive
② Zero
③ Negative
} Initial velocity



2.2 Cases (I) ($b^2 - 4ac > 0$)

The polynomial $\phi(r)$ have two distinct real roots $r_1 > r_2$. Then, $\phi(D) = (D - r_1)(D - r_2)$ and re-write the equation as:

$$L(y) = (D - r_1)(D - r_2)y = 0.$$

letting

$$z = (D - r_2)y,$$

Case II. Critical Damping

dengan $c^2 = 4mk,$

$$\beta = 0,\ \lambda_1 = \lambda_2 = -\alpha.$$

$$y(t) = (c_1 + c_2 t)e^{-\alpha t}.$$

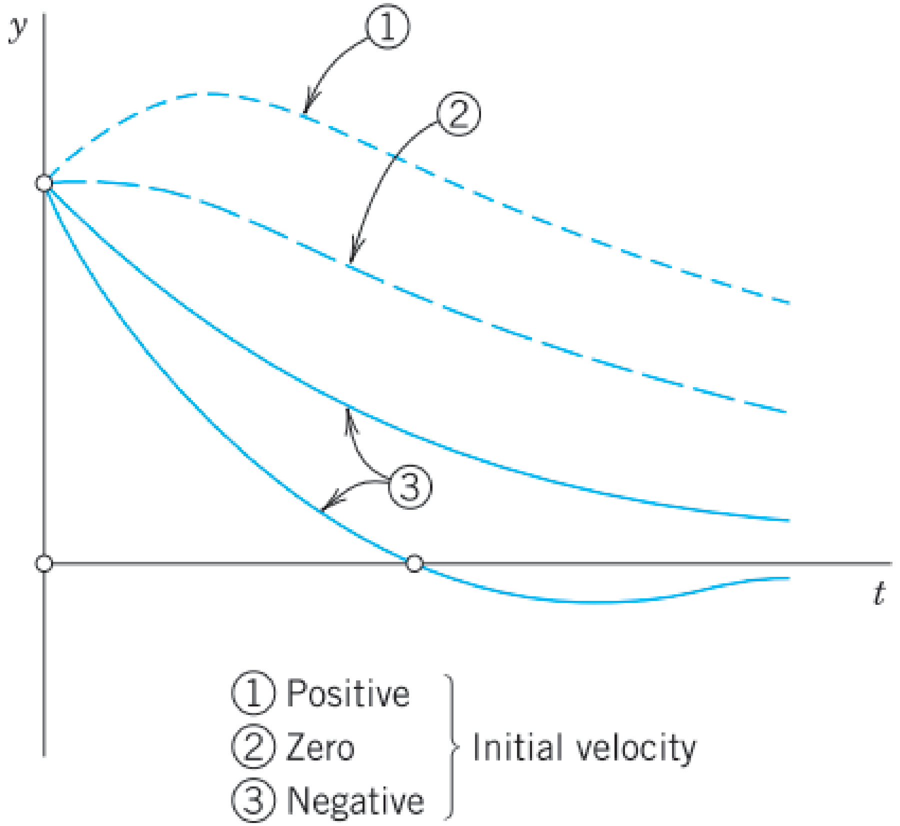


Fig. 38. Critical damping [see (8)]

Case III. Underdamping

$c^2 < 4mk.$

$$\beta = i\omega^* \qquad \omega^* = \frac{1}{2m}\sqrt{4mk - c^2} = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}} \qquad (>0).$$

dimana

$$\lambda_1 = -\alpha + i\omega^*, \qquad \lambda_2 = -\alpha - i\omega^*$$

$\alpha = c/(2m),$

$$y(t) = e^{-\alpha t}(A \cos \omega^* t + B \sin \omega^* t) = Ce^{-\alpha t} \cos (\omega^* t - \delta)$$

$$C^2 = A^2 + B^2 \qquad \tan \delta = B/A$$

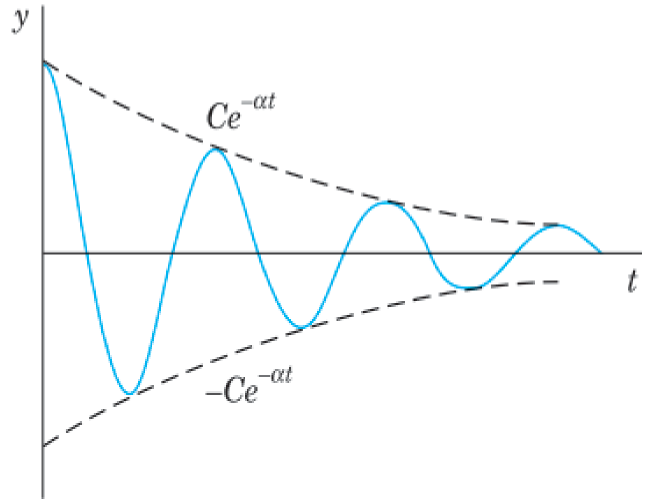


Fig. 39. Damped oscillation in Case III [see (10)]

Contoh

$$\text{(I) } c = 100 \text{ kg/sec}, \qquad \text{(II) } c = 60 \text{ kg/sec}, \qquad \text{(III) } c = 10 \text{ kg/sec}.$$

Nilai parameter sistem

$m = 10 \qquad k = 90,$

$$10y'' + 100y' + 90y = 0, \qquad y(0) = 0.16 \text{ [meter]}, \qquad y'(0) = 0.$$

dengan $10\lambda^2 + 100\lambda + 90 = 10(\lambda + 9)(\lambda + 1) = 0.$ akar -9 & $-1.$

$y = c_1 e^{-9t} + c_2 e^{-t}.$ Juga $y' = -9c_1 e^{-9t} - c_2 e^{-t}.$

$c_1 + c_2 = 0.16, -9c_1 - c_2 = 0.$ $c_1 = -0.02, c_2 = 0.18.$

$$y = -0.02e^{-9t} + 0.18e^{-t}.$$

(II) The model is as before, with $c = 60$ instead of 100. The characteristic equation now has the form $10\lambda^2 + 60\lambda + 90 = 10(\lambda + 3)^2 = 0$. It has the double root -3. Hence the corresponding general solution is

$$y = (c_1 + c_2 t)e^{-3t}. \qquad \text{We also need} \qquad y' = (c_2 - 3c_1 - 3c_2 t)e^{-3t}.$$

The initial conditions give $y(0) = c_1 = 0.16$, $y'(0) = c_2 - 3c_1 = 0$, $c_2 = 0.48$. Hence in the critical case the solution is

$$y = (0.16 + 0.48t)e^{-3t}.$$

It is always positive and decreases to 0 in a monotone fashion.

(III) The model now is $10y'' + 10y' + 90y = 0$. Since $c = 10$ is smaller than the critical c, we shall get oscillations. The characteristic equation is $10\lambda^2 + 10\lambda + 90 = 10[(\lambda + \frac{1}{2})^2 + 9 - \frac{1}{4}] = 0$. It has the complex roots [see (4) in Sec. 2.2 with $a = 1$ and $b = 9$]

$$\lambda = -0.5 \pm \sqrt{0.5^2 - 9} = -0.5 \pm 2.96i.$$

This gives the general solution

$$y = e^{-0.5t}(A \cos 2.96t + B \sin 2.96t).$$

Thus $y(0) = A = 0.16$. We also need the derivative

$$y' = e^{-0.5t}(-0.5A \cos 2.96t - 0.5B \sin 2.96t - 2.96A \sin 2.96t + 2.96B \cos 2.96t).$$

Hence $y'(0) = -0.5A + 2.96B = 0, \quad B = 0.5A/2.96 = 0.027$. This gives the solution

$$y = e^{-0.5t}(0.16 \cos 2.96t + 0.027 \sin 2.96t) = 0.162e^{-0.5t} \cos (2.96t - 0.17).$$

We see that these damped oscillations have a smaller frequency than the harmonic oscillations in Example 1 by about 1% (since 2.96 is smaller than 3.00 by about 1%). Their amplitude goes to zero. See Fig. 40. ■

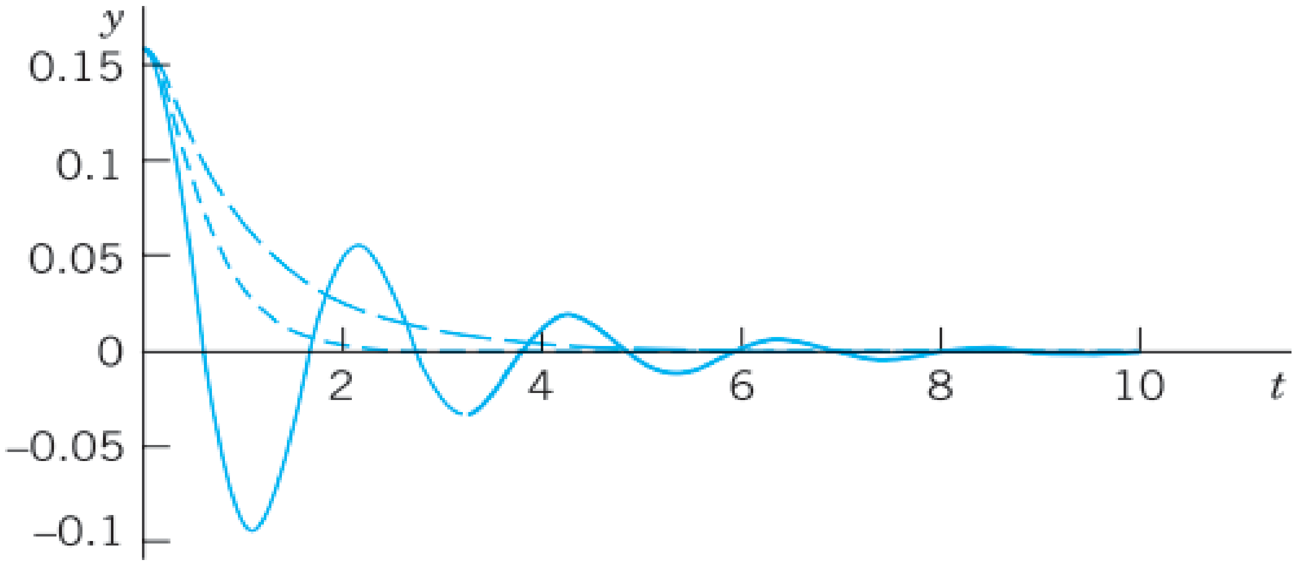


Fig. 40. The three solutions in Example 2

Tugas Teamwork

1 **TEAM PROJECT. Harmonic Motions of Similar Models.** The ***unifying power of mathematical methods*** results to a large extent from the fact that different physical (or other) systems may have the same or very similar models. Illustrate this for the following three systems

(a) Pendulum clock. A clock has a 1-meter pendulum. The clock ticks once for each time the pendulum completes a full swing, returning to its original position. How many times a minute does the clock tick?

(b) Flat spring (Fig. 45). The harmonic oscillations of a flat spring with a body attached at one end and horizontally clamped at the other are also governed by (3). Find its motions, assuming that the body weighs 8 nt (about 1.8 lb), the system has its static equilibrium 1 cm below the horizontal line, and we let it start from this position with initial velocity 10 cm/sec.

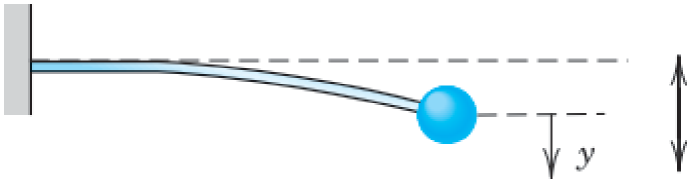


Fig. 45. Flat spring

2 $$9\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 1 = 0$$

3 $$\frac{d^2\theta}{dt^2} + 2k\frac{d\theta}{dt} + n^2\theta = 0$$

Tugas 3 dikumpulkan paling lambat 18 Oktober 2020, jam 24/00 mell MyClassroom

- Catat semua Informasi tambahan dari perkuliahan – online

Terimakasih