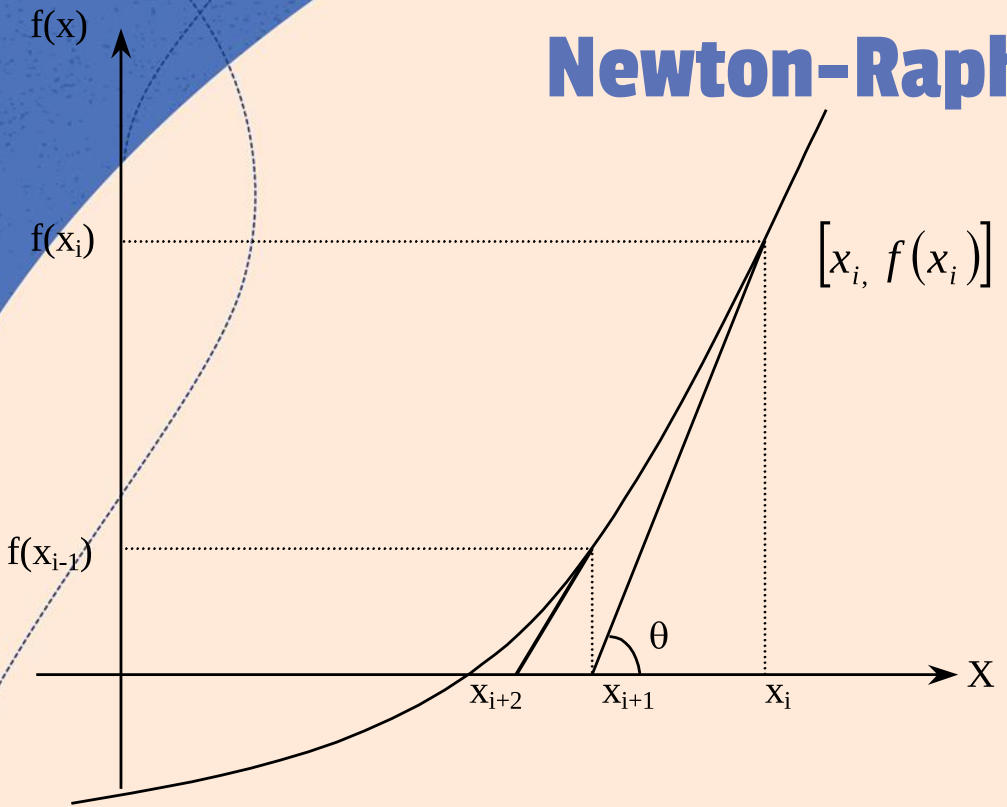
Newton-Raphson
&
SECANT METHOD

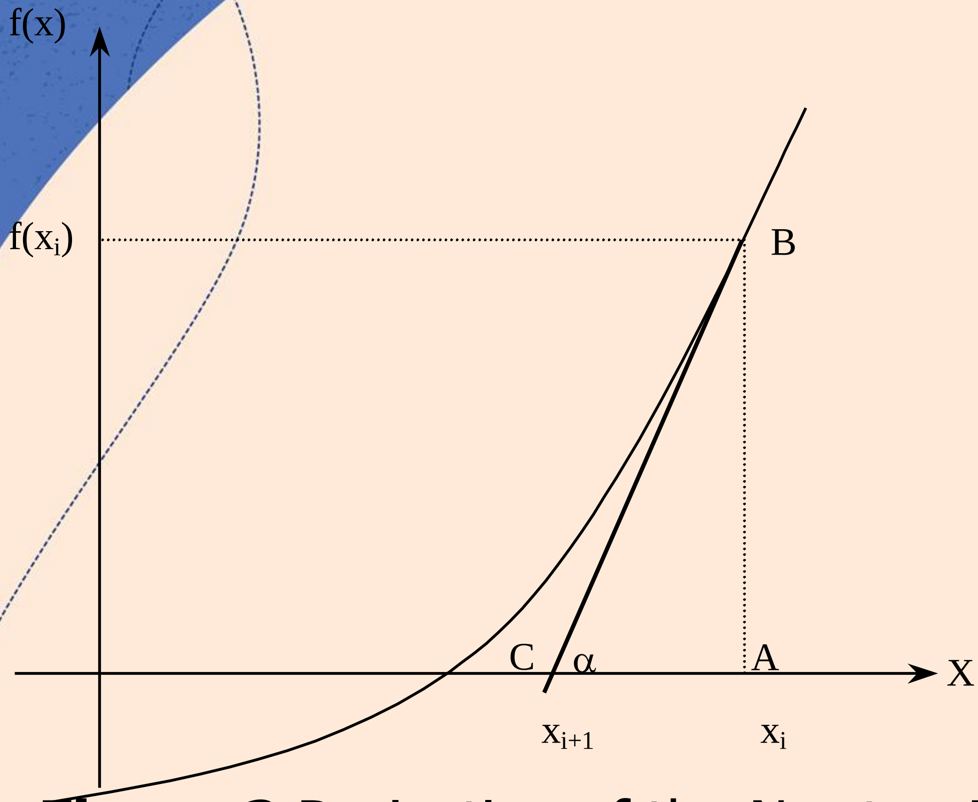
Newton-Raphson Method



$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Figure 1 Geometrical illustration of the Newton-Raphson method.

Derivation



$$\tan(\alpha) = \frac{AB}{AC}$$

$$f'(x_i) = \frac{f(x_i)}{x_i - x_{i+1}}$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Figure 2 Derivation of the Newton-Raphson method.

Algorithm for Newton-Raphson Method

Step 1

Evaluate $f'(x)$ symbolically.

Step 2

Use an initial guess of the root, x_i, to estimate the new value of the root, x_{i+1}, as

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

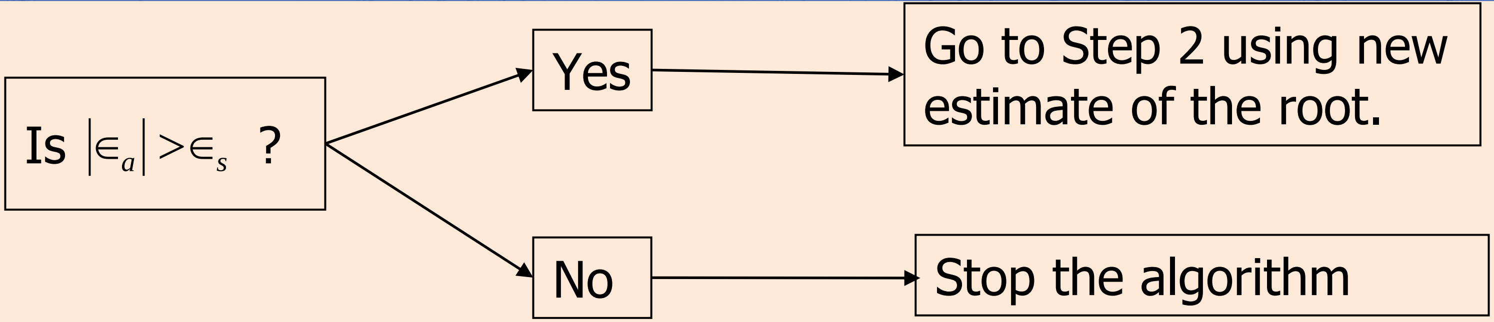
Step 3

Find the absolute relative approximate error $|\in_a|$ as

$$|\in_a| = \left|\frac{x_{i+1} - x_i}{x_{i+1}}\right| \times 100$$

Step 4

Compare the absolute relative approximate error with the pre-specified relative error tolerance .



Also, check if the number of iterations has exceeded the maximum number of iterations allowed. If so, one needs to terminate the algorithm and notify the user.

Newton-Raphson

Square Root

Set up a Newton iteration for computing the square root x of a given positive number c and apply it to $c = 2$.

Solution. We have $x = \sqrt{c}$, hence $f(x) = x^2 - c = 0$, $f'(x) = 2x$, and (5) takes the form

$$x_{n+1} = x_n - \frac{x_n^{\ 2} - c}{2x_n} = \frac{1}{2}\left(x_n + \frac{c}{x_n}\right).$$

For $c = 2$, choosing $x_0 = 1$, we obtain

$$x_1 = 1.500\,000, \qquad x_2 = 1.416\,667, \qquad x_3 = 1.414\,216, \qquad x_4 = 1.414\,214, \cdots.$$

x_4 is exact to 6D. ■

Iteration for a Transcendental Equation

Find the positive solution of $2 \sin x = x$.

Solution. Setting $f(x) = x - 2 \sin x$, we have $f'(x) = 1 - 2 \cos x$, and (5) gives

$$x_{n+1} = x_n - \frac{x_n - 2 \sin x_n}{1 - 2 \cos x_n} = \frac{2(\sin x_n - x_n \cos x_n)}{1 - 2 \cos x_n} = \frac{N_n}{D_n}.$$

From the graph of f we conclude that the solution is near $x_0 = 2$. We compute:

n	x_n	N_n	D_n	x_{n+1}
0	2.00000	3.48318	1.83229	1.90100
1	1.90100	3.12470	1.64847	1.89552
2	1.89552	3.10500	1.63809	1.89550
3	1.89550	3.10493	1.63806	1.89549

$x_4 = 1.89549$ is exact to 5D since the solution to 6D is 1.895 494.

Newton's Method Applied to an Algebraic Equation

Apply Newton's method to the equation $f(x) = x^3 + x - 1 = 0$.

Solution. From (5) we have

$$x_{n+1} = x_n - \frac{x_n^{\ 3} + x_n - 1}{3x_n^{\ 2} + 1} = \frac{2x_n^{\ 3} + 1}{3x_n^{\ 2} + 1}.$$

Starting from $x_0 = 1$, we obtain

$$x_1 = 0.750\,000, \qquad x_2 = 0.686\,047, \qquad x_3 = 0.682\,340, \qquad x_4 = 0.682\,328, \cdots$$

where x_4 has the error $-1 \cdot 10^{-6}$. A comparison with Example 2 shows that the present convergence is much more rapid. This may motivate the concept of the *order of an iteration process*, to be discussed next. ■

Convergence of Newton's Method

In Newton's method, $g(x) = x - f(x)/f'(x)$. By differentiation,

$$
\begin{aligned}
g'(x) &= 1 - \frac{f'(x)^2 - f(x)f''(x)}{f'(x)^2} \\
&= \frac{f(x)f''(x)}{f'(x)^2} .
\end{aligned}
\tag{8}
$$

Since $f(s) = 0$, this shows that also $g'(s) = 0$. Hence Newton's method is at least of second order. If we differentiate again and set $x = s$, we find that

$$g''(s) = \frac{f''(s)}{f'(s)} \tag{8*}$$

which will not be zero in general. This proves

SECANT METHOD

Secant Method – Derivation

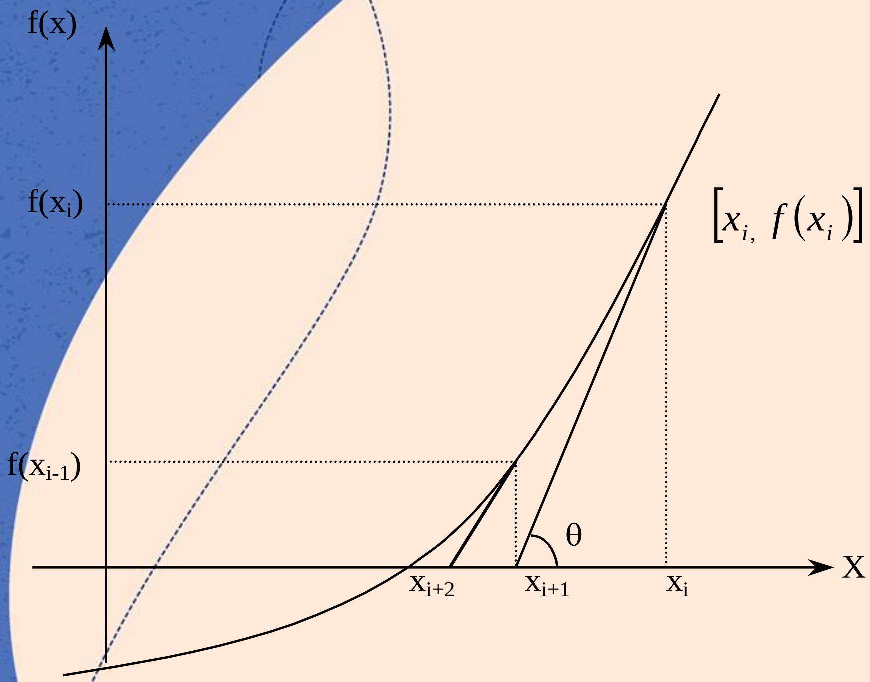


Figure 1 Geometrical illustration of the Newton-Raphson method.

Newton's Method

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \quad (1)$$

Approximate the derivative

$$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} \quad (2)$$

Substituting Equation (2) into Equation (1) gives the Secant method

$$x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$$

Secant Method – Derivation

The secant method can also be derived from geometry:

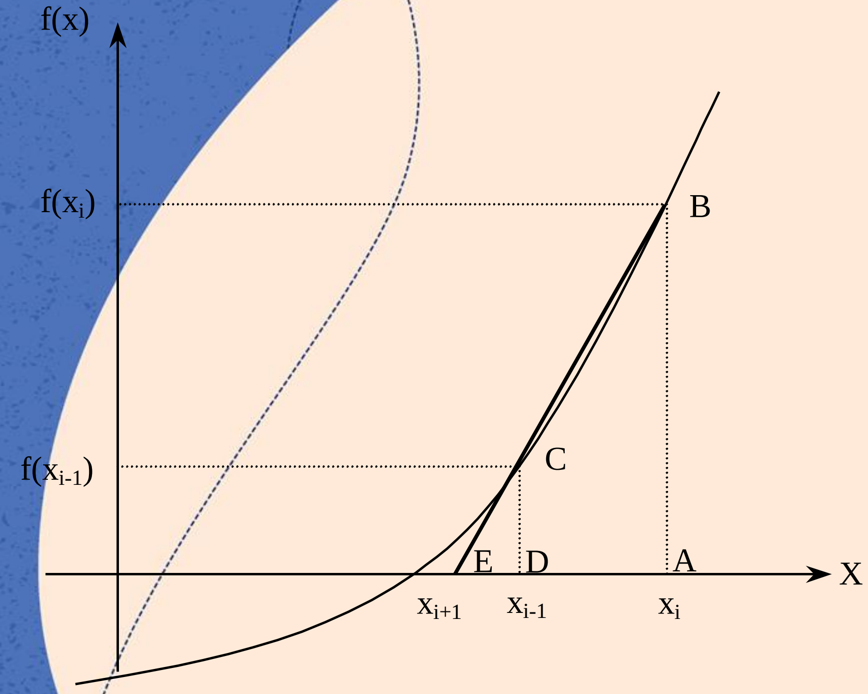


Figure 2 Geometrical representation of the Secant method.

The Geometric Similar Triangles

$$\frac{AB}{AE} = \frac{DC}{DE}$$

can be written as

$$\frac{f(x_i)}{x_i - x_{i+1}} = \frac{f(x_{i-1})}{x_{i-1} - x_{i+1}}$$

On rearranging, the secant method is given as

$$x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$$

Algorithm Secant Method

Step 1

Calculate the next estimate of the root from two initial guesses

$$x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$$

Find the absolute relative approximate error

$$|\in_a| = \left| \frac{x_{i+1} - x_i}{x_{i+1}} \right| \times 100$$

Step 2

Find if the absolute relative approximate error is greater than the prespecified relative error tolerance.

If so, go back to step 1, else stop the algorithm.

Also check if the number of iterations has exceeded the maximum number of iterations.

Secant Method

Find the positive solution of $f(x) = x - 2\sin x = 0$ by the secant method, starting from $x_0 = 2$, $x_1 = 1.9$.

Solution. Here, (10) is

$$x_{n+1} = x_n - \frac{(x_n - 2\sin x_n)(x_n - x_{n-1})}{x_n - x_{n-1} + 2(\sin x_{n-1} - \sin x_n)} = x_n - \frac{N_n}{D_n}.$$

Numerical values are:

n	x_{n-1}	x_n	N_n	D_n	$x_{n+1} - x_n$
1	2.000 000	1.900 000	−0.000 740	−0.174 005	−0.004 253
2	1.900 000	1.895 747	−0.000 002	−0.006 986	−0.000 252
3	1.895 747	1.895 494	0		0

$x_3 = 1.895\,494$ is exact to 6D. See Example 4. ■