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PENGENDALIAN – SISTEM NONLINIER

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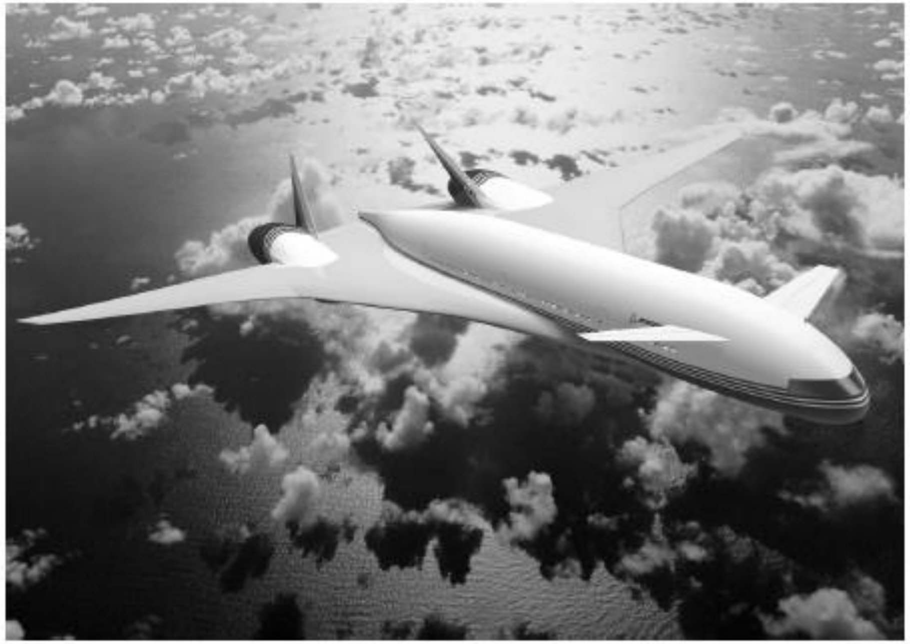
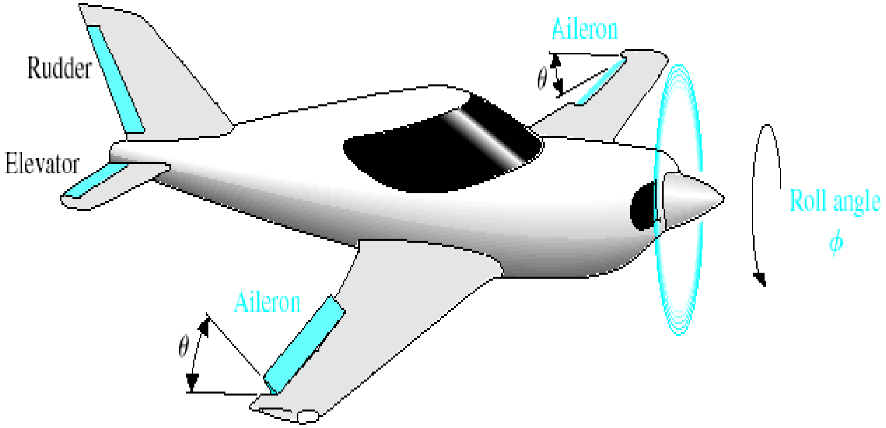
Chap7 Nonlinear Control system

7.1 Introduction

7.2 Describing function

7.3 Method of the phase locus





Chap7 Nonlinear Control system

7.1 Introduction

7.1.1 What is the nonlinearity ?

7.1.2 What is the nonlinear control system?

7.1.3 The typical nonlinearities.

7.1.4 The speciality of the nonlinear systems

7.1.5 Analysis method of the nonlinear systems

7.1 Introduction

7.1.1 What is the nonlinearity ?

The “output” varying is not proportional to the “input” varying for a device.

The characteristic of the nonlinear device can not be described by the linear differential equation.

Types of the nonlinearity:

(1) Essential nonlinearity

The nonlinearity $y=f(x)$ can not be expressed as the Talor series expansion in all x.

(2) Nonessential nonlinearity

The $y=f(x)$ can be expressed as the Talor series expan-sion in all x.

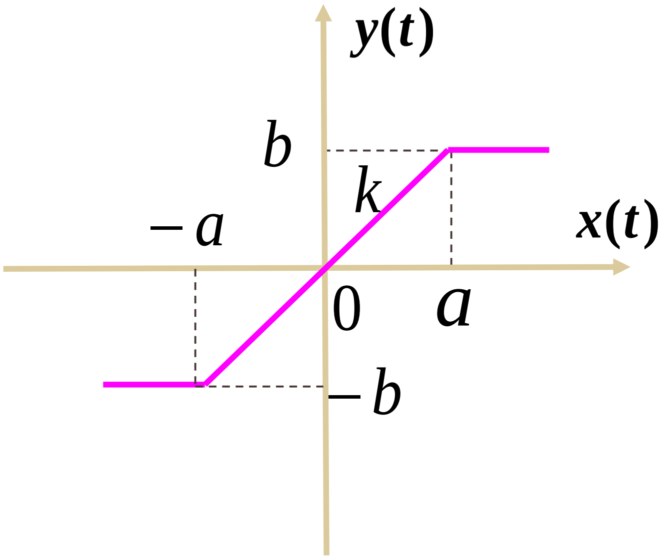
7.1 Introduction

7.1. 2 What is the nonlinear control system?

If a control system include one or more nonlinear characteristic element or link , the system is named as the nonlinear control system.

7.1.3 The typical nonlinearities

(1) Saturation nonlinearity



Mathematical description

$$y(t)=\begin{cases} kx(t) & |x(t)|\le a \\ ka\cdot sign\; x(t) & |x(t)|>a \end{cases}$$

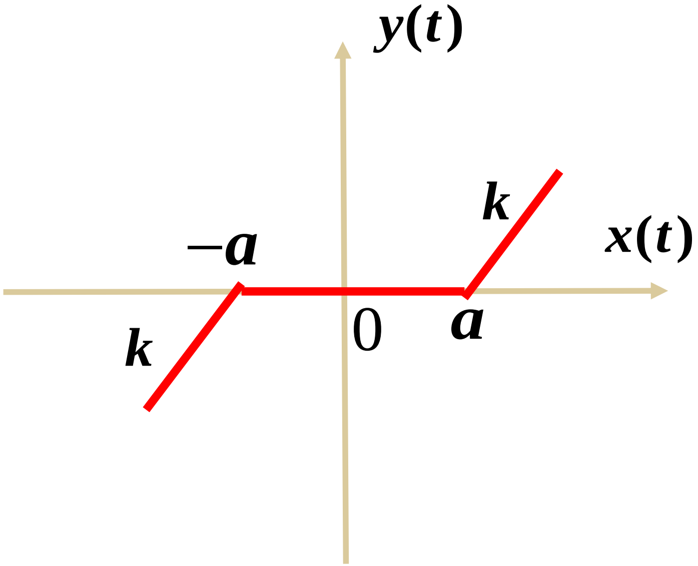
a – linear zone width; k – slope of the linear characteristic;

$$sign\; x(t)=\begin{cases} +1 & x(t)>0 \\ -1 & x(t)<0 \end{cases}$$

7.1.3 The typical nonlinearities

Actual examples: saturation characteristic of the amplifier;valve journey; power limit etc.

(2) Dead zone nonlinearity



Mathematical description:

$$y(t)=\begin{cases}0 & |x(t)|\le a\\ k[x(t)-a\operatorname{sing} x(t)] & |x(t)|>a\end{cases}$$

a – *dead zone width*; k – *slope of the linear output*;

$$\mathit{sign}\ x(t)=\begin{cases}+1 & x(t)>0\\ -1 & x(t)<0\end{cases}$$

Actual examples: **Insensitive zone of the measure system;**
Turn on characteristic of the diode etc.

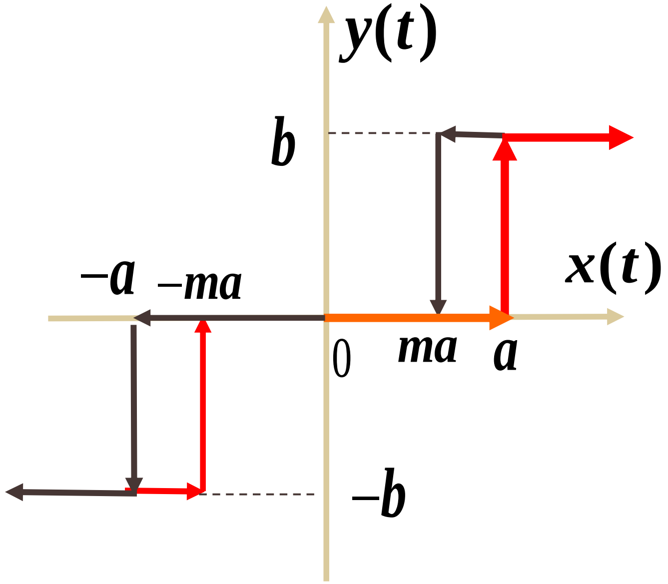
7.1.3 The typical nonlinearities

(3) Relay nonlinearity

Mathematical description:

$$y(t)=\begin{cases}-b & x(t)<-ma \\ 0 & -ma<x(t)<a \\ b & x(t)>a \end{cases}\quad \dot{x}(t)>0$$

$$\begin{cases} b & x(t)>ma \\ 0 & -a<x(t)<ma \\ -b & x(t)<-a\end{cases}\quad \dot{x}(t)<0$$



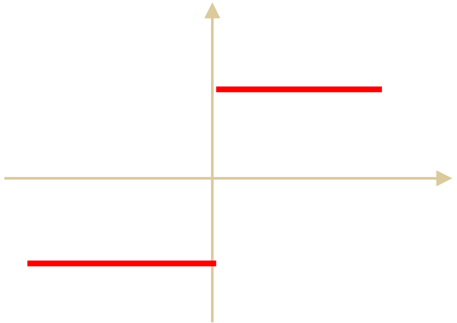
a → relay attracting voltage; ma → relay release voltage

b → saturation output

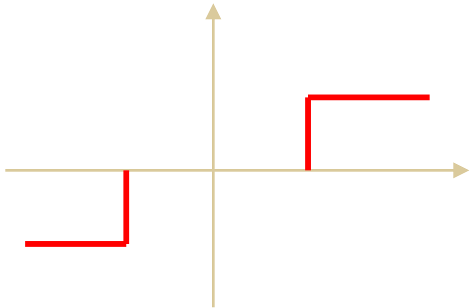
Actual examples: relay, switch etc.

Several special relay nonlinearity:

7.1.3 The typical nonlinearities



Ideal relay nonlinearity



Approximate relay nonlinearity (m=1)

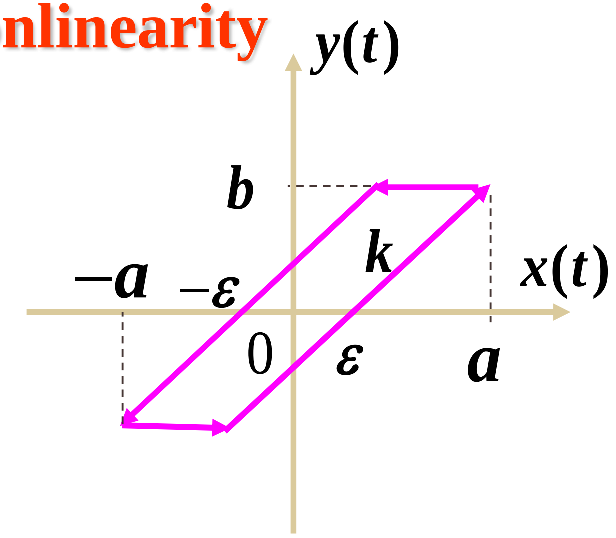
(4) Backlash hysteresis loop(clearance) nonlinearity

***Mathematical description*:**

$$y(t)=\begin{cases} k\left[x(t)-\varepsilon\right] & \dot{y}(t)>0 \\ k\left[x(t)+\varepsilon\right] & \dot{y}(t)\le 0 \\ b\,\mathrm{sign}\,x(t) & \dot{y}(t)=0 \end{cases}$$

$2\varepsilon \rightarrow$ *backlash width*

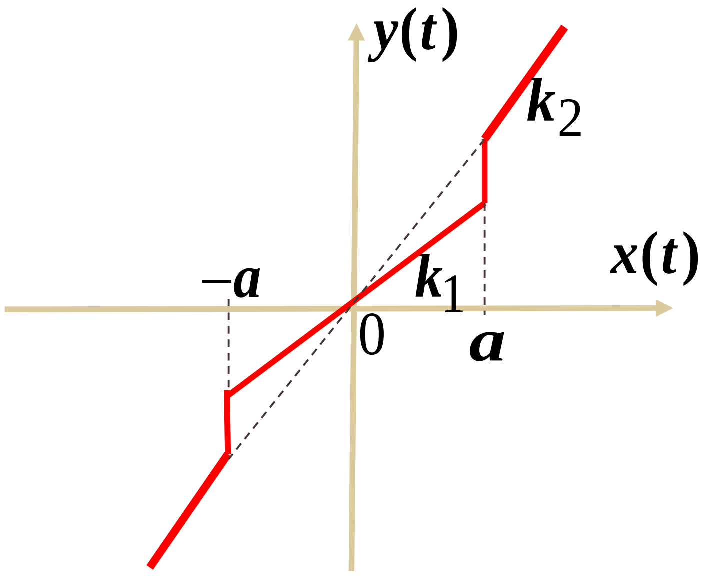
$k \rightarrow$ *slope of the backlash characteristic*



Actual example: gear backlash

7.1.3 The typical nonlinearities

(5) changeable gain nonlinearity



Mathematical description:

$$y(t)=\begin{cases} k_1 x(t) & |x(t)|<a \\ k_2 x(t) & |x(t)|>a \end{cases}$$

$a \rightarrow$ *change point*

$k_1\ k_2 \rightarrow$ *slope of the changeable gain characteristic*

7.1.4 The characteristics of the nonlinear systems (distinguishing features with linear system)

	Linear system characteristics	Nonlinear system characteristics
1	Satisfy superposition theorem.	Not satisfy superposition theorem.
2	Stability is only related to the system parameters.	Stability is related to system input, initial state, parameters, structure etc.
3	Have two kind of work states: stable and unstable.	Have stable, unstable and self-oscillation.
4	The form of the output is the same as input.	The form of the output is different from the input.

7.1.4 The analysis and design methods of the nonlinear systems

① Phase plane method
② Describing function method } **Classical**
③ Computer and intelligence → **modern**

7.2 Describing function method of the nonlinear system analysis

Four items:

1. What is the describing function?
2. How to get the describing function? ← (modeling)
3. How to analyze a nonlinear system by describing function? ← (analysis and design)
4. Attentions and development

7.2.1 What is the describing function?

(Put forwarded by P.J.Daniel, In 1940)

1. Basic idea

For the nonlinear system

7.2.1 What is the describing function?

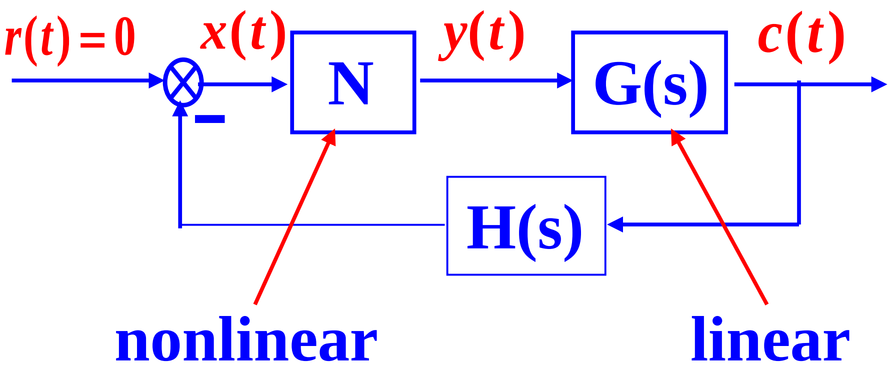


Fig.7.7 Typical structure of the nonlinear systems

If : $x(t)=X\sin\omega t \Longrightarrow$ a sinusoidal input,

$y(t)$, maybe it is not a sinusoidal but a periodic function, can be expressed as a Fourier series:

$$y(t)=A_0+\sum_{n=1}^{\infty}(A_n\cos n\omega t+B_n\sin n\omega t)$$

$$=A_0+\sum_{n=1}^{\infty}Y_n\sin(n\omega t+\varphi_n)$$

7.2.1 What is the describing function?

$$y(t) = A_0 + \sum_{n=1}^{\infty}(A_n \cos n\omega t + B_n \sin n\omega t)$$

$$= A_0 + \sum_{n=1}^{\infty} Y_n \sin(n\omega t + \varphi_n)$$

$$Y_n = \sqrt{A_n^2 + B_n^2}, \quad \varphi_n = arctg\frac{A_n}{B_n}$$

$$A_0 = \frac{1}{2\pi}\int_0^{2\pi} y(t)d(\omega t)$$

$$A_n = \frac{1}{\pi}\int_0^{2\pi} y(t)\cos n\omega t d(\omega t)$$

$$B_n = \frac{1}{\pi}\int_0^{2\pi} y(t)\sin n\omega t d(\omega t)$$

Discuss:

i) For the symmetry nonlinearity: $A_0 = 0$, and

ii) the harmonic of $y(t)$ could be neglected, then:

$$y(t) \approx Y_1 \sin(\omega t + \varphi_1) \Longrightarrow$$

output frequency is equal to input frequency approximately.

7.2.1 What is the describing function?

It means:

We can describe the nonlinear components by the frequency response like as that we did in chapter 5.

So we have:

2. Definition of the describing function

The describing function $N(X)$ of the nonlinear element is: the complex ratio of the fundamental component of the output $y(t)$ and the sinusoidal input $x(t)$, that is:

$$For\ \ x(t) = X \sin\omega t,$$

$$y(t) \approx A_1 \cos\omega t + B_1 \sin\omega t$$

$$= Y_1 \sin(\omega t + \varphi_1) \Longrightarrow N(X) = \frac{Y_1 e^{j\varphi_1}}{X}$$

Here:

7.2.1 What is the describing function?

$$\begin{matrix} Y_1 = \sqrt{A_1^2 + B_1^2} \\ \varphi_1 = arctg\dfrac{A_1}{B_1} \end{matrix} \longleftarrow \begin{cases} A_1 = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} y(t)\cos\omega t d(\omega t) \\ B_1 = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} y(t)\sin\omega t d(\omega t) \end{cases}$$

Because the describing function actually is the linearized "frequency response" → "harmonic linearization",

we can analyze the nonlinear systems like as that we did in chapter 5.

7.2.2 How to get the describing function?

1. Steps

(1) Input a sinusoid signal $x(t)$ to the nonlinear elements:

$$x(t) = X\sin\omega t$$

7.2.2 How to get the describing function?

(2) **Solve $y(t)$ and obtain the fundamental component of $y(t)$.**

(3) **Calculate the describing function $N(X)$ according to following formula:**

$$\left.\begin{aligned} A_1 &= \frac{1}{\pi}\int_0^{2\pi} y(t)\cos\omega t\, d(\omega t) \\ B_1 &= \frac{1}{\pi}\int_0^{2\pi} y(t)\sin\omega t\, d(\omega t) \end{aligned}\right\} \left.\begin{aligned} Y_1 &= \sqrt{A_1^2 + B_1^2} \\ \varphi_1 &= arctg\frac{A_1}{B_1} \end{aligned}\right\} y(t) \approx Y_1 \sin(\omega t + \varphi_1)$$

$$\downarrow$$

$$N(X) = \frac{Y_1 e^{j\varphi_1}}{X}$$

2. Examples

7.2.2 How to get the describing function?

Example 7.1

The mathematical description of a nonlinear device is:

$$y = \frac{1}{2}x + \frac{1}{4}x^3$$

Determine the describing function of the device.

Solution

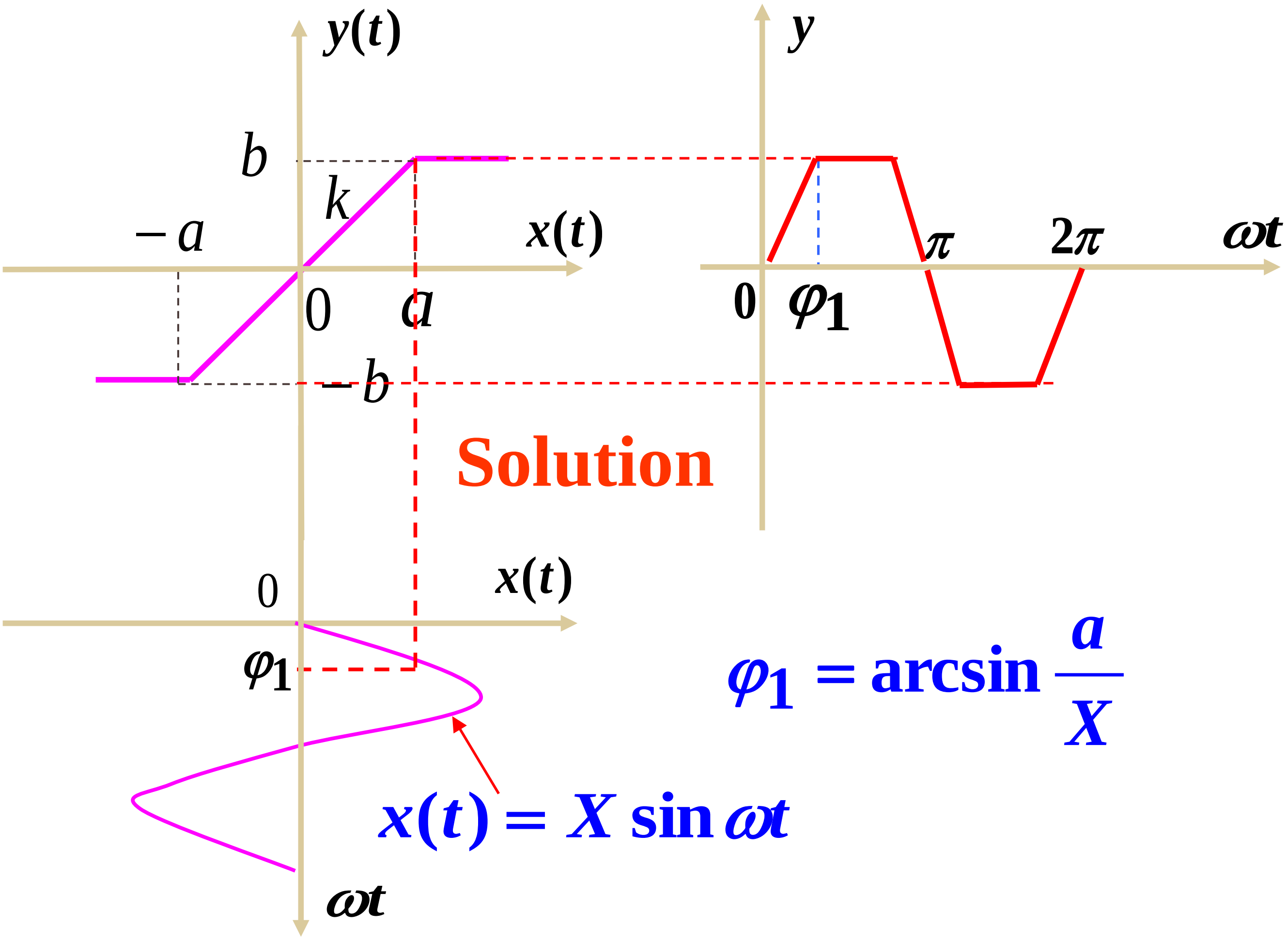
$$y(t) = \frac{1}{2}x + \frac{1}{4}x^3 \Big|_{x=X\sin\omega t}$$

$$= (\frac{1}{2}X + \frac{3}{16}X^3)\sin\omega t - \frac{1}{16}X^3 \sin 3\omega t$$

$$y_1(t) = (\frac{1}{2}X + \frac{3}{16}X^3)\sin\omega t \Longrightarrow \mathrm{N(X)} = \frac{\dot{\mathrm{Y}}_1}{\dot{X}} = \frac{1}{2} + \frac{3}{16}X^2$$

7.2.2 How to get the describing function?

Example 7.2 **Determine the describing function of the saturation nonlinearity.**

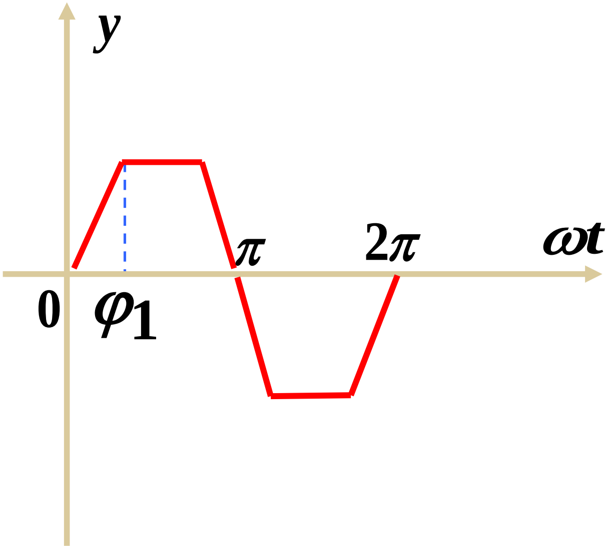


Solution

$$\varphi_1 = \arcsin\frac{a}{X}$$

$$x(t) = X\sin\omega t$$

7.2.2 How to get the describing function?



$$y(t)=\begin{cases}kX\sin\omega t & 0\le\omega t\le\varphi_1\\ kX & \varphi_1\le\omega t\le\dfrac{\pi}{2}\end{cases}$$

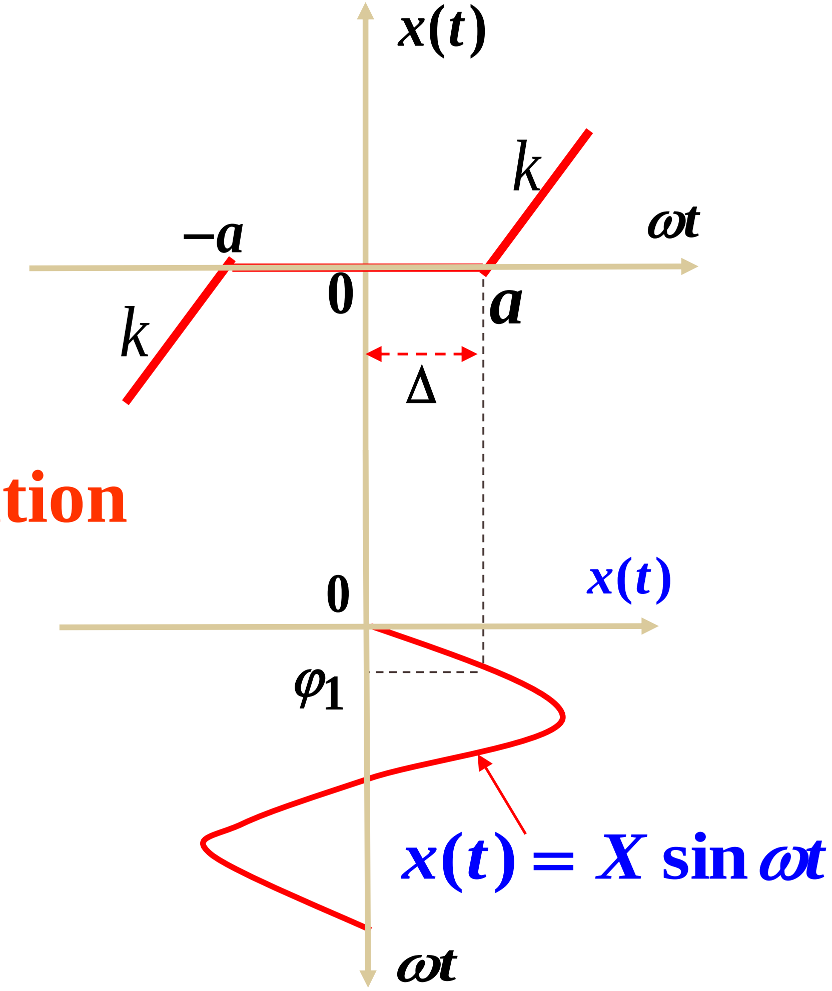
$$A_1=0 \qquad B_1=\frac{1}{\pi}\int_0^{2\pi}y(t)\sin\omega t d(\omega t)$$

$$=\frac{2kX}{\pi}\left[\arcsin\frac{a}{X}+\frac{a}{X}\sqrt{1-(\frac{a}{X})^2}\right]$$

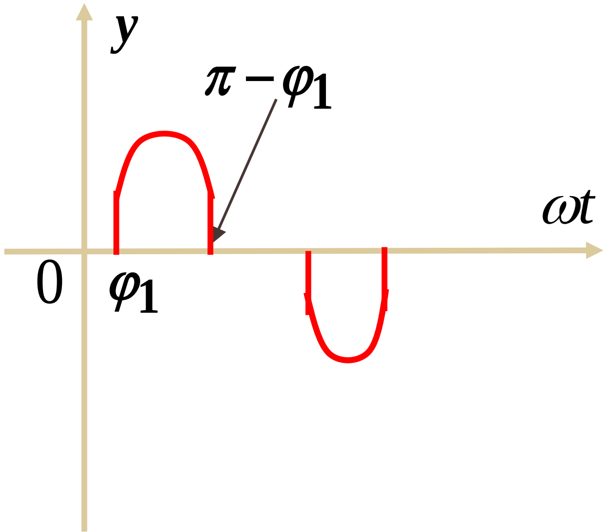
$$N(X)=\frac{B_1}{\mathrm{X}}=\frac{2k}{\pi}\left[\arcsin\frac{a}{X}+\frac{a}{X}\sqrt{1-(\frac{a}{X})^2}\right] \qquad X>a$$

7.2.2 How to get the describing function?

Example 7.3 **Determine the describing function of the dead zone nonlinearity.**



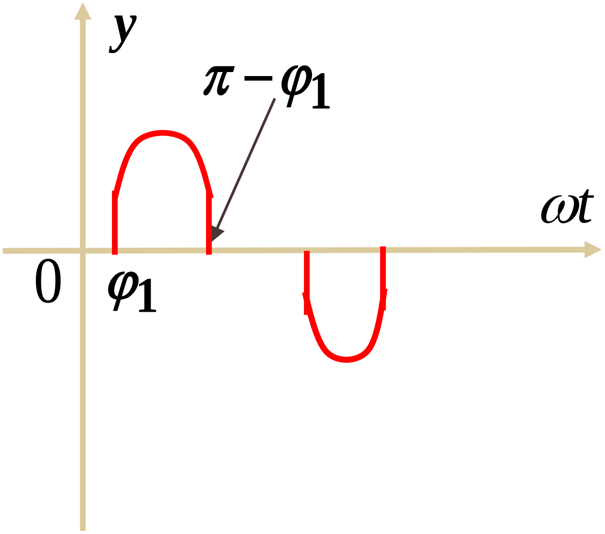
solution



$$\varphi_1 = \arcsin\frac{\Delta}{X}$$

7.2.2 How to get the describing function?

$$\mathrm{y(t)} = \begin{cases} 0 & 0 \le \omega t \le \varphi_1 \\ k(X\sin\omega t - \Delta) & \varphi_1 \le \omega t \le \dfrac{\pi}{2} \end{cases}$$



$$A_1 = 0$$

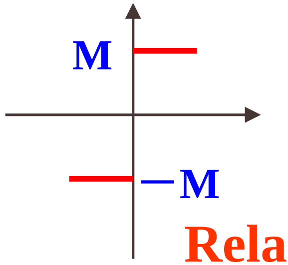
$$B_1 = \frac{1}{\pi}\int_0^{2\pi} y(t)\sin\omega t d(\omega t)$$

$$= \frac{2kX}{\pi}\left[\frac{\pi}{2} - \arcsin\frac{\Delta}{X} - \frac{\Delta}{X}\sqrt{1-(\frac{\Delta}{X})^2}\right] \qquad (\mathrm{X} \ge \Delta)$$

$$N(X) = \frac{B_1}{X} = \frac{2k}{\pi}\left[\frac{\pi}{2} - \arcsin\frac{\Delta}{X} - \frac{\Delta}{X}\sqrt{1-(\frac{\Delta}{X})^2}\right] \qquad (\mathrm{X} \ge \Delta)$$

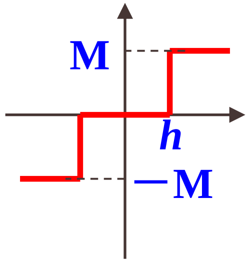
7.2.2 How to get the describing function?

3. The describing function of some typical nonlinearity

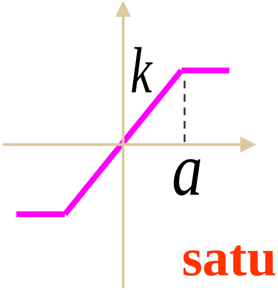


$$N(X)=\frac{4M}{\pi X}$$

Relay

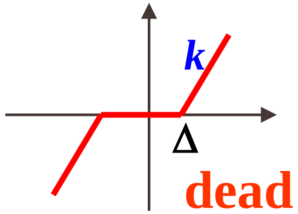


$$N(X)=\frac{4M}{\pi X}\sqrt{1-(\frac{h}{X})^2}$$



$$N(X)=\frac{2k}{\pi}\left[\arcsin\frac{a}{X}+\frac{a}{X}\sqrt{1-(\frac{a}{X})^2}\right]$$

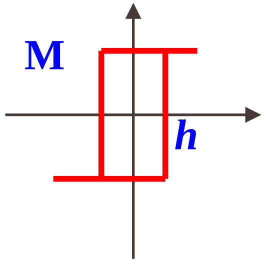
saturation



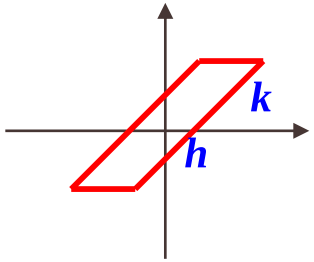
$$N(X)=\frac{2k}{\pi}\left[\frac{\pi}{2}-\arcsin\frac{\Delta}{X}-\frac{\Delta}{X}\sqrt{1-(\frac{\Delta}{X})^2}\right]$$

dead zone

3. The describing function of some typical nonlinearity



$$N(X)=\frac{4M}{\pi X}\sqrt{1-(\frac{h}{X})^2}-j\frac{4M}{\pi X^2}$$



backlash hysteresis

$$N(X)=\frac{k}{\pi}\left[\begin{array}{l}\frac{\pi}{2}+\arcsin(1-\frac{2h}{X})\\ +2(1-\frac{2h}{X})\sqrt{\frac{h}{X}(1-\frac{h}{X})}\end{array}\right]$$

$$+j\frac{4kh}{\pi X}(\frac{h}{X}-1)$$

7.2.2 How to get the describing function?

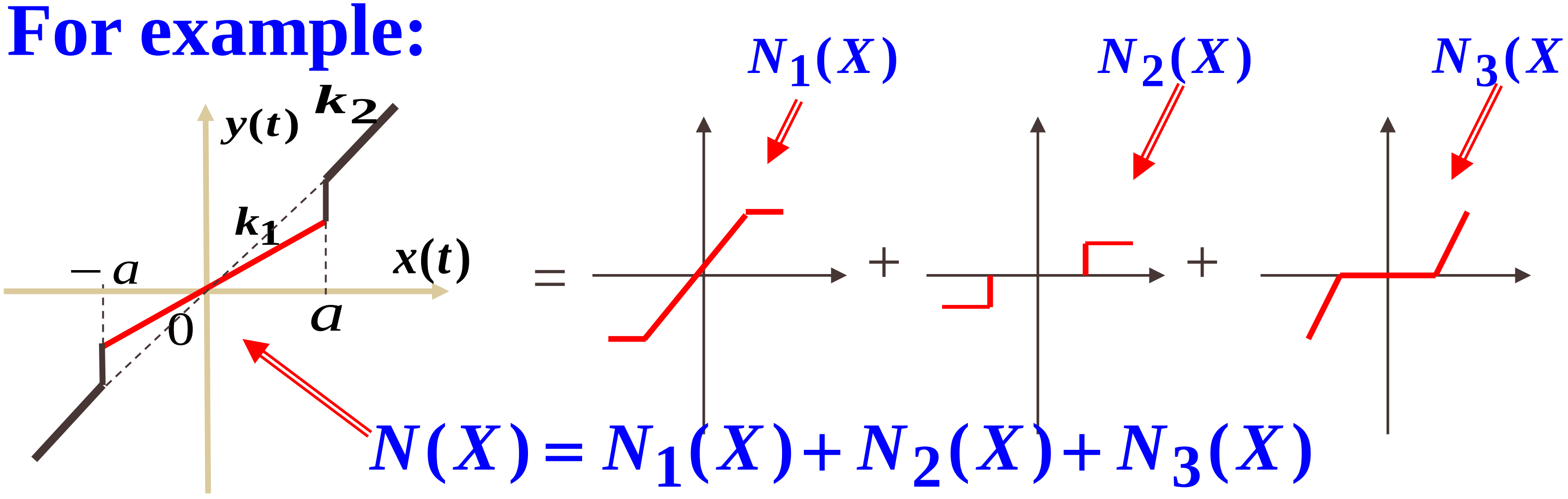
4. characters

(1) For the “single value” nonlinearity the describing function must be a “real number”.

such as the dead zone, saturation and the ideal relay nonlinearity etc.

(2) The describing function satisfy the superposition principle (nonlinearity not).

For example:



$$N(X) = N_1(X) + N_2(X) + N_3(X)$$

7.2.3 Stability analysis of the nonlinear system by describing function

1. Review of Nyquist criterion

For the linear system:

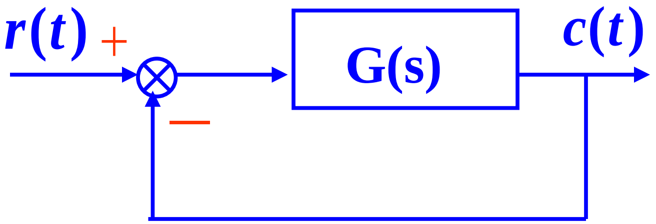


Fig.7.2.3.1

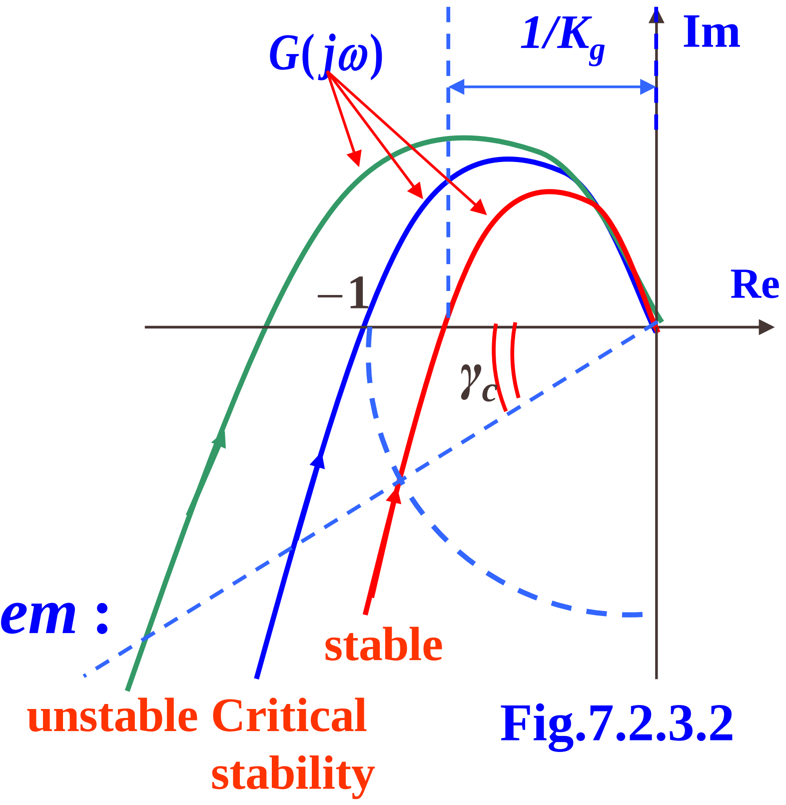


Fig.7.2.3.2

The characteristic equation of the system :

$$1+G(j\omega)=0$$

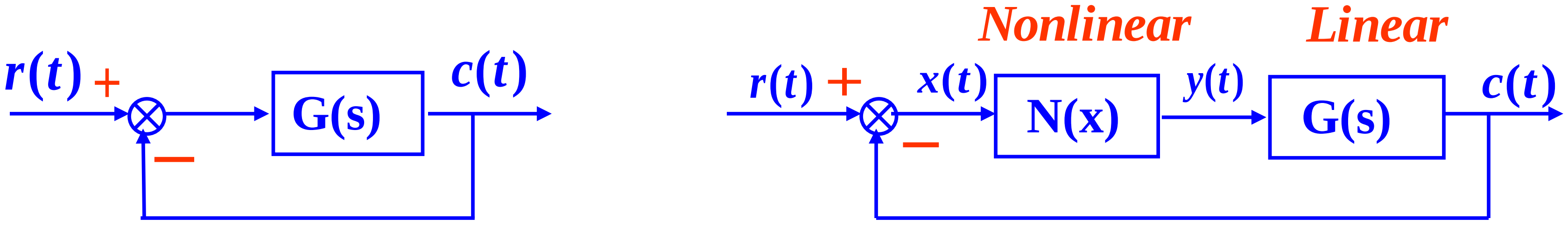
$$\Rightarrow G(j\omega)=-1+j0$$

If $G(s)$ is a minimum phase transfer function, the necessary and sufficient condition of the stable system is :

$G(j\omega)$ *does not circle the point* $(-1, j\omega)$

7.2.3 Stability analysis of the nonlinear system by describing function

2. Compare the nonlinear system with the linear system



Linear system Fig.7.2.3.1 nonlinear system

Transfer function of the system:

$$\phi(j\omega)=\frac{C(j\omega)}{R(j\omega)}=\frac{G(j\omega)}{1+G(j\omega)}$$

$$\phi(j\omega)=\frac{C(j\omega)}{R(j\omega)}=\frac{N(X)G(j\omega)}{1+N(X)G(j\omega)}$$

Characteristic equation:

$$1+G(j\omega)=0$$

$$\Rightarrow G(j\omega)=-1$$

$$1+N(X)G(j\omega)=0$$

$$\Rightarrow G(j\omega)=-\frac{1}{N(X)}$$

In the $G(j\omega)$ plane

A point

A curve

Because the describing function N(X) actually is a linearized frequency response, we can expand the Nyquist criterion to the nonliear system :

3. Stability analysis of the nonlinear system

(For example the minimum phase system)

(1) $G(j\omega)$ **don't circle the** $-\dfrac{1}{N(X)}$ **curve, the nonlinear system is stable;**

(2) $G(j\omega)$ **circle the** $-\dfrac{1}{N(X)}$ **curve, the nonlinear system is unstable;**

(3) $G(j\omega)$ **intersect with the** $-\dfrac{1}{N(X)}$ **curve, there will be a self-oscillation in the nonlinear system.**

compare with linear system

(1) $G(j\omega)$ **don't circle the point** $(-1, j\omega)$**, the system is stable;**

(2) $G(j\omega)$ **circle the point** $(-1, j\omega)$**, the system is unstable;**

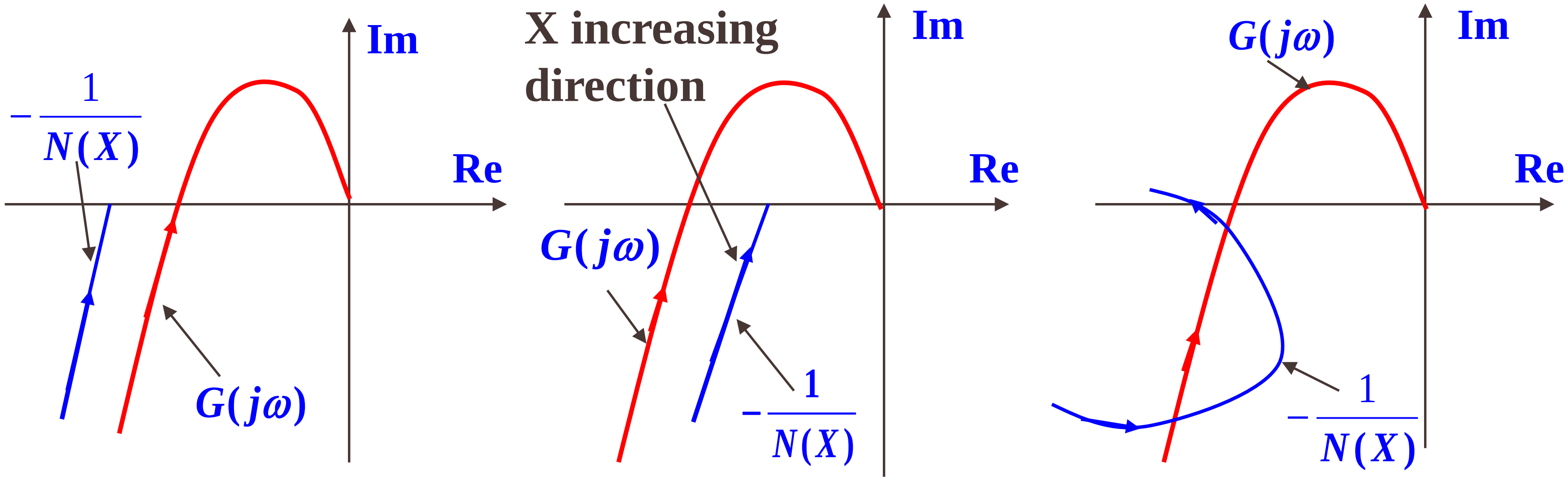
(3) $G(j\omega)$ **intersect with the point** $(-1, j\omega)$**, the system is in the critical stability.**

3. Stability analysis of the nonlinear system

(For example the minimum phase system)

$-\dfrac{1}{N(X)}$: Negative inverse describing function.

Graphical explanation is shown as following:



$G(j\omega)$ do not circle －1/N(X) (stable)

$G(j\omega)$ circle －1/N(X) (unstable)

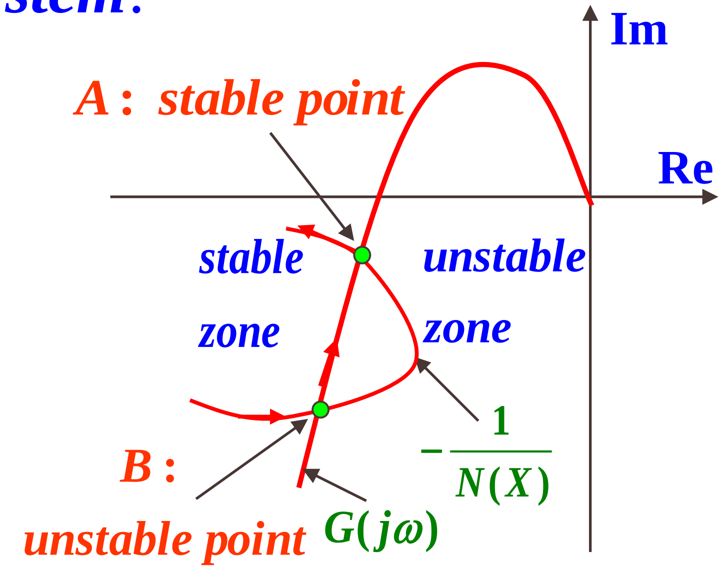
$G(j\omega)$ Intersect with －1/N(A) (self-oscillation)

7.2.3 Stability analysis of the nonlinear system by describing function

4. Self-oscillation of the nonlinear system

A special motion of the nonlinear system:

System will be at a continuous oscillation, which has a constant amplitude and frequency, when the system come under a light disturbance.



Self-oscillation

Corresponding to the intersection point of G(jω) with $-\dfrac{1}{N(X)}$ **:**

B: *unstable self-oscillation point→ –1/N(X)***enter unstable zone from stable zone.**

A: *stable self-oscillation point→ –1/N(X)* **enter stable zone from unstable zone.**

Example: (a graduate examination)

A nonlinear system is shown in Fig.7.2.3.4. The describing function of the Relay nonlinearity is $\frac{4}{\pi X}$, $G(s)=\frac{K}{s(5s+1)(10s+1)}$

1) Determine the system's stability.

2) Determine K and oscilation frequency ω when self-oscilation amplitude is $X=\frac{1}{\pi}$.

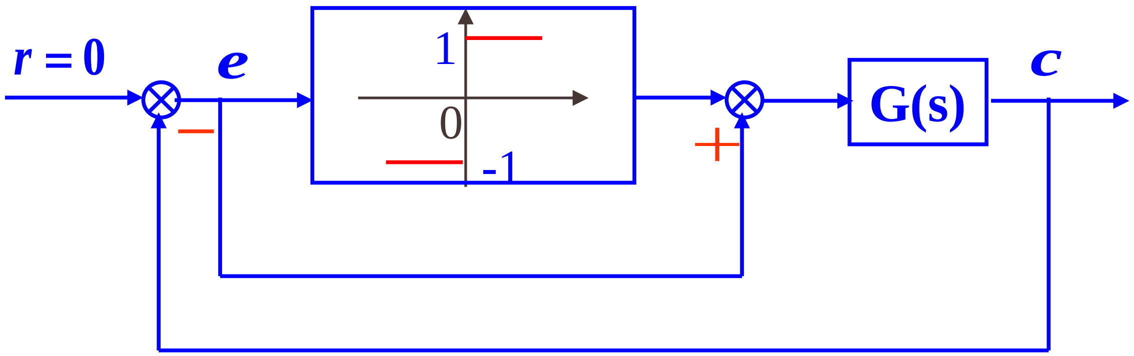


Fig.7.2.3.4

Solution:

The system is equivalent to the Fig.7.2.3.5.

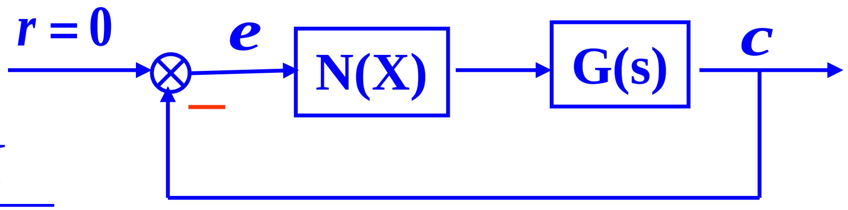


Fig.7.2.3.5

$$N(X)=1+\frac{4}{\pi X}\Rightarrow-\frac{1}{N(X)}=-\frac{\pi X}{\pi X+4}$$

Graphical explanation is shown as Fig.7.2.3.6

$$G(j\omega)=\frac{K}{j\omega(j5\omega+1)(j10\omega+1)}$$

$$=\frac{K}{j\omega\left[(1-50\omega^2)+j15\omega\right]}$$

$$G(j\omega)\Big|_{\omega=\sqrt{\frac{1}{50}}\approx 0.14}=-\frac{10}{3}K$$

$$-\frac{1}{N(X)}=-\frac{\pi X}{\pi X+4}=\begin{cases}0 & X=0\\ 1 & X=\infty\end{cases}$$

Exercise: for this example, if:

$$G(s)=\frac{K}{s(10s+1)^2}$$

$$\omega=\sqrt{\frac{1}{50}}\approx 0.14$$

$$G(j\omega)=-\frac{10}{3}K$$

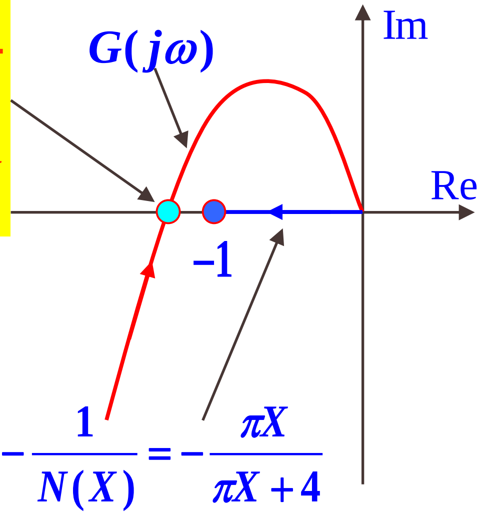


Fig.7.2.3.6

(1) **Stability analysis:** $\begin{cases}K>\frac{3}{10}, & \text{unstable.}\\ K\le\frac{3}{10}, & \text{self-oscillation}\end{cases}$

(2) $$-\frac{1}{N(X)}=-\frac{\pi X}{\pi X+4}\Big|_{X=\frac{1}{\pi}}=-\frac{1}{5}=G(j\omega)\Rightarrow\omega=\sqrt{\frac{1}{50}}\approx 0.14,\quad K=\frac{3}{50}$$

7.2.4 Attentions and development

1. Attentions

(1) Using the describing function to analyze the nonlinear system, the Linear parts of the system must be provided with a good charact-eristic of the low-pass filter→so that the harmonics produced by the nonlinear element can be neglected.

(2) Generally the describing function method can only be used for analyzing the stability and self-oscillation of the nonlinear systems, not the stead-state error and transient specifications.

2. development

Modern analysis and design method of the nonlinear systems:
Computer simulation and intelligent design.

7.3 Phase plane method

It is a kind of graphic method to solve first and second order differential equation, put forward by Poincare In 1885.

Four items:

1. What is the Phase plane？
2. How to plot the Phase plane ？
3. How to analyze the nonlinear systems by the Phase plane method.
4. Attentions and development.

7.3.1 What is the Phase plane

For a second-order time-invariable system:

$$\ddot{x} = f(x, \dot{x})$$

$f(x, \dot{x})$ is a linear or nonlnear function of $x(t)$ and $\dot{x}(\mathrm{t})$.

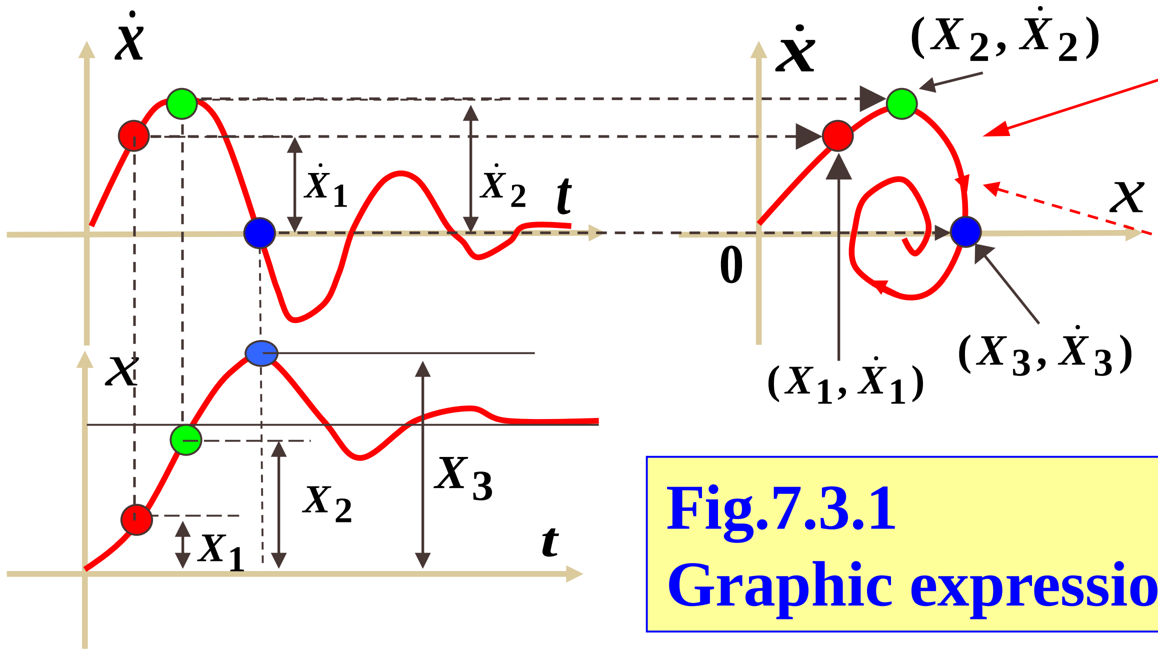
The solution of $\ddot{x} = f(x, \dot{x})$ can be expressed by the form of the relation curve between $x(t)$ *and* $\dot{x}(t)$

7.3.1 What is the Phase plane

In the rectangular coordinate plane constituted by :

x - axis$_{[=x(t)]}$ and y - axis$_{[=\dot{x}(t)]}$

- **the plane $\Rightarrow$ the phase plane**
- **$x(t)$ and $\dot{x}(t)$ $\Rightarrow$ the phase plane variables (state variable).**
- **relation curve between $x(t)$ *and* $\dot{x}(t) \Rightarrow$ phase trajectory.**



Phase plane graph

Increasing *t* direction

Always to be clockwise

Fig.7.3.1
Graphic expression

7.3.2. plotting method of the phase locus

1. Analytic method

For the system : $\ddot{x} + f(x, \dot{x}) = 0$

Because : $\ddot{x} = \frac{d\dot{x}}{dt} = \frac{d\dot{x}}{dx}\frac{dx}{dt} = \dot{x}\frac{d\dot{x}}{dx} = x_2\frac{dx_2}{dx}$

Make : $x = x_1 \quad \dot{x} = x_2$ *we have :* $x_2\frac{dx_2}{dx_1} = -f(x_1, x_2)$

If $f(x_1, x_2) \overset{\text{can be}}{\underset{\text{decomposed}}{=}} f_1(x_1) \cdot f_2(x_2) \Rightarrow \frac{x_2}{f_2(x_2)}dx_2 = -f_1(x_1)dx_1$

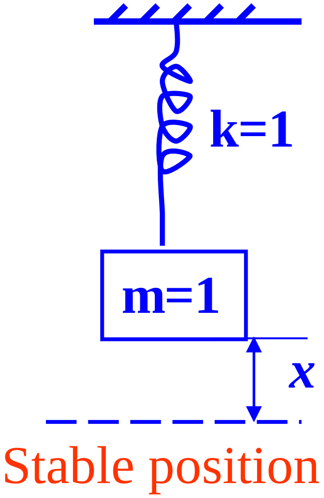
Then : $\underbrace{\int \frac{x_2}{f_2(x_2)}dx_2}_{\Downarrow} = \underbrace{-\int f_1(x_1)dx_1}_{\Downarrow}$

we have : $F_2(x_2) = F_1(x_1)$

The relationship between $x_2(=\dot{x})$ *and* $x_1(=x)$ *is obtained.*

1. Analytic method

Example 7.3.1:



Spring - mass motion system : $m\ddot{x} + Kx = 0$

$m \rightarrow$ **mass,** $K \rightarrow$ **spring constant.**

If initial condition $x(0) = x_0, \dot{x}(0) = 0$ **,**

plot the phase loci.

Solution: $m\ddot{x} + Kx\big|_{m=1, K=1} = 0 \Rightarrow \ddot{x} + x = 0$

$$then: \dot{x}\frac{d\dot{x}}{dx} = -x \Rightarrow \int \dot{x}d\dot{x} = -\int xdx$$

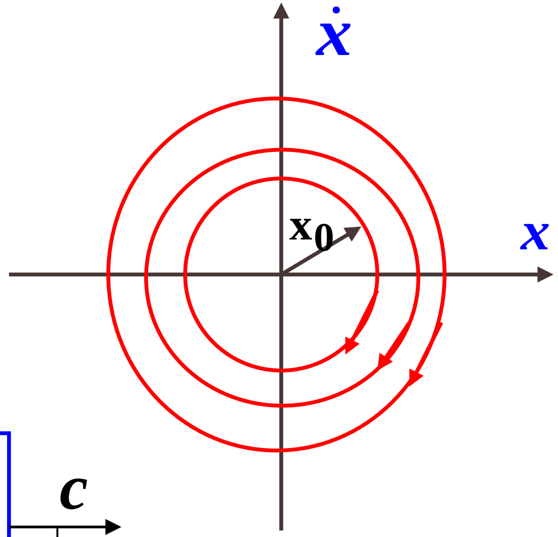
$$\Rightarrow \frac{1}{2}\left[\dot{x}^2 - \dot{x}^2(0)\right] = -\frac{1}{2}\left[x^2 - x^2(0)\right]$$

Because $x(0) = x_0, \dot{x}(0) = 0 \Rightarrow \dot{x}^2 + x^2 = x_0^2$

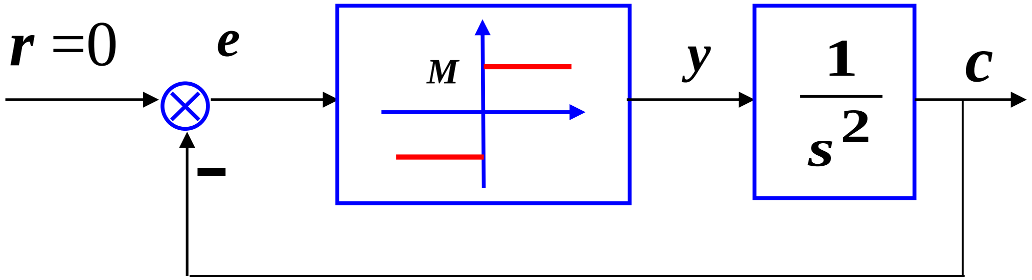
The phase trajectory is a circle, x_0 **is the radius.**

7.3.2. plotting method of the phase locus

For different x_0 the phase loci are a tuft of concentric circles shown in following figure.



Example 7.3.2: **For the system:**



Plot the phase loci of the system:

Solution: $$\textit{Because}: Y(s)\cdot\frac{1}{s^2}=C(s)\rightarrow Y(s)=s^2C(s)\Rightarrow y=\ddot{c}$$

$$\textit{So we have}: \frac{d^2c}{dt^2}=y=\begin{cases}M & r-c>0\\ -M & r-c<0\end{cases}=M\text{sign}(r-c)$$

$$make: \quad c = x_1, \dot{c} = x_2$$

$$then: \quad \left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \ddot{c} = Msign(r - x_1) \end{aligned}\right\} \Rightarrow \frac{dx_2}{dx_1} = \frac{Msign(r - x_1)}{x_2}$$

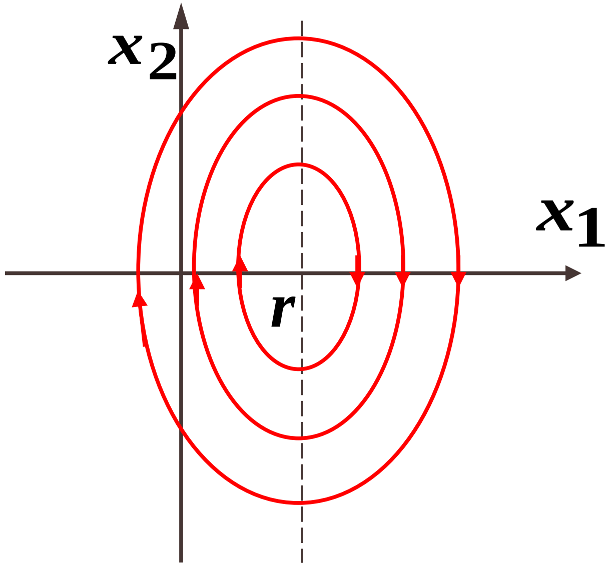
$$that\ is \int_{x_2(0)}^{x_2} x_2 dx_2 = \int_{x_1(0)}^{x_1} Msign(r - x_1) dx_1$$

$$when \quad x_1 < r:$$

$$we\ have \quad x_2^2 = 2Mx_1 - 2Mx_1(0) + x_2^2(0)$$

$$when \quad x_1 > r:$$

$$we\ have \quad x_2^2 = -2Mx_1 + 2Mx_1(0) + x_2^2(0)$$



The phase loci of the system is shown in following figure→self-oscillation.

7.3.2. plotting method of the phase locus

2. Graphic method---isoclinal method

For the systems : $\ddot{x}+f(x,\dot{x})=0 \Rightarrow \dot{x}\dfrac{d\dot{x}}{dx}+f(x,\dot{x})=0$

make : $x_1=x, x_2=\dot{x}$ then : $x_2\dfrac{dx_2}{dx_1}=-f(x_1,x_2)$

$$\Rightarrow \frac{dx_2}{dx_1}=-\frac{f(x_1,x_2)}{x_2}$$

make : $\dfrac{dx_2}{dx_1}=\alpha \Rightarrow \underset{\downarrow}{\alpha}=-\dfrac{f(x_1,x_2)}{x_2}$ → isocline equation

α : the slope of the phase loci

Example 7.3.3: Spring - mass motion system : $\ddot{x}+x=0$

plot the phase loci by the isoclinal method.

solution

In terms of : $\ddot{x} + x = 0$ we have : $\frac{d\dot{x}}{dx} = \frac{-x}{\dot{x}}$

make $\frac{d\dot{x}}{dx} = \alpha$ then : $\dot{x} = -\frac{1}{\alpha} x$

The isocline is the beelines passing the coordinate origin and with slope $-\frac{1}{\alpha}$.

The $-\frac{1}{\alpha}$ values are shown in following table for different α

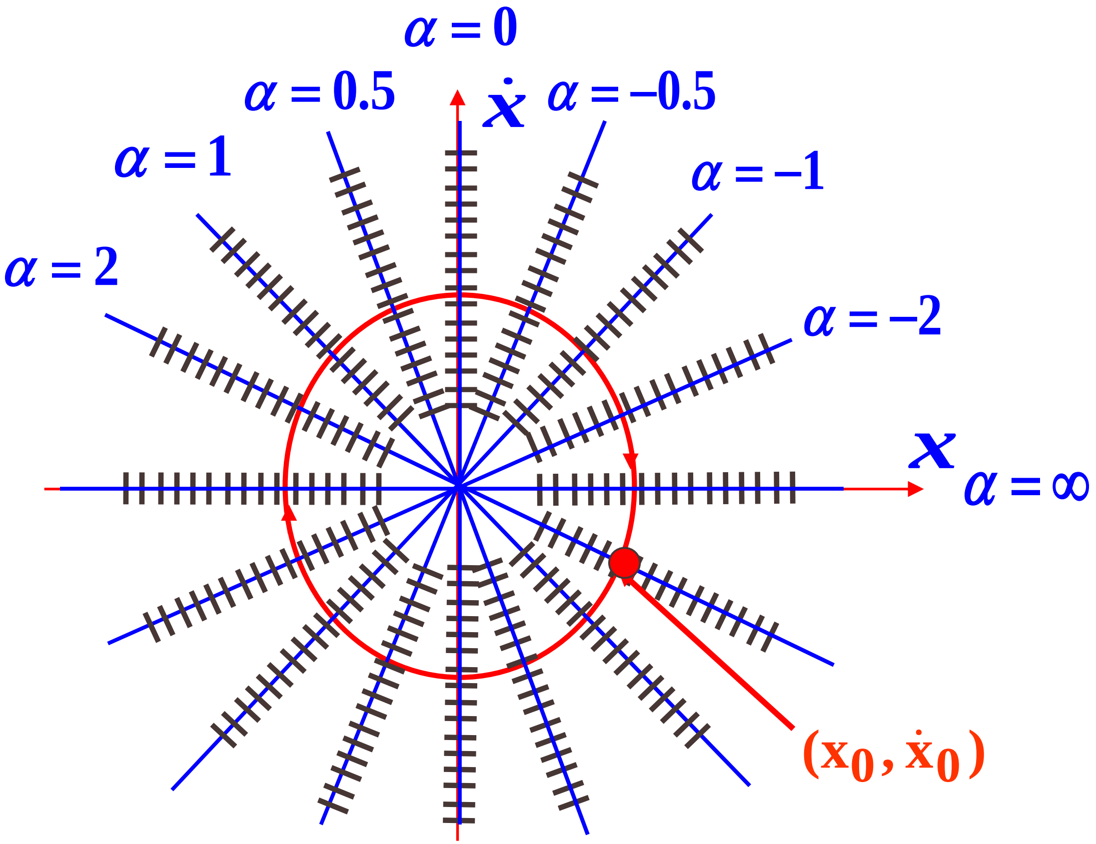
α	∞	2	1	0.5	0	-0.5	-1	-2	$-\infty$
$-\frac{1}{\alpha}$	0	-0.5	-1	-2	$-\infty$	2	1	0.5	0

We can plot the isoclinals like as following figure:

(1) Plot the isoclinals for different α.

(2) Plot the corresponding tangents of the phase loci in each isoclinals. α is the slope of the phase loci.

(3) Plot the phase loci starting at the initial states $(x_0, \dot{x}_0)$.



7.3.2. plotting method of the phase locus

Attentions:

1) x - axis and $\dot{x}$ shoud have the same scale.

2) The direction of the phase loci always are clockwise :

For $\dot{x} > 0$: from left to right with x increasing ;

For $\dot{x} < 0$: from right to left with x decreasing .

3) The slope of the phase loci through x - axis is $\alpha = \infty$,

so the phase loci intersect x - axis uprightly.

4) apply the symmetry of the phase locus to reduce work .

For the symmetry about $\dot{x}$ – axis: $f(x, \dot{x}) = -f(-x, \dot{x})$

For the symmetry about $\dot{x}$ – axis: $f(x, \dot{x}) = f(x, -\dot{x})$

For the symmetry about origin: $f(x, \dot{x}) = -f(-x, -\dot{x})$

7.3.3 Analysis of the phase plane

1. Singularity points of the phase locus

(1) singularit y points

$$\text{For}: \ \ddot{x} = \dot{x}\frac{d\dot{x}}{dx} = -f(x,\dot{x}), \quad \text{slope}: \alpha = \frac{d\dot{x}}{dx} = -\frac{f(x,\dot{x})}{\dot{x}}$$

if $f(x,\dot{x}) = 0$ and $\dot{x} = 0$ at the same time, then:

$$\frac{d\dot{x}}{dx} = \frac{0}{0} \rightarrow \text{indefinite slope.}$$

$\Rightarrow$ singularity point.

$*$ There are infinite phase loci going to or going off the singularity point because of the indefinite slpoe .

$*$ The singularity points are the balance points of the nonliear systems because of $\dot{x} = 0$ at the points.

1. Singularity points of the phase locus

(2) Types of the singularity points

The linearized nonlinear differential equation in the neighborhood of the singularity point can be expressed :

$\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2 x = 0$ Characteristic equation: $s^2 + 2\xi\omega_n s + \omega_n^2 = 0$

$s_{1,2} = -\xi\omega_n \pm \sqrt{\xi^2\omega_n^2 - \omega_n^2}$

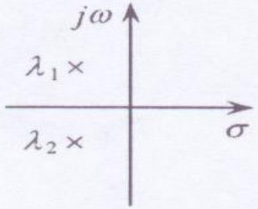
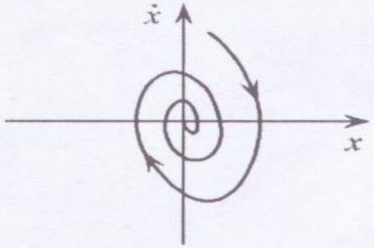
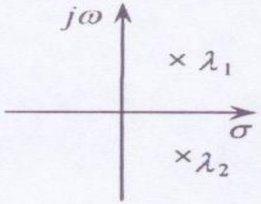
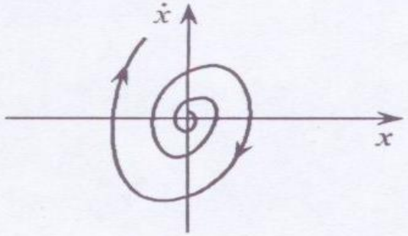
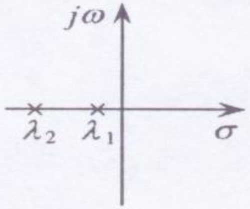
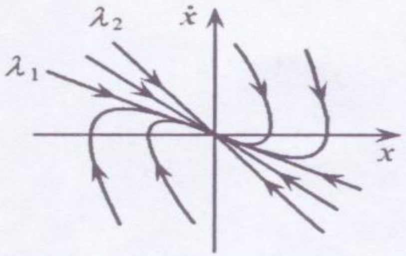
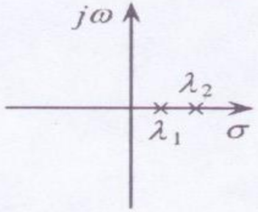
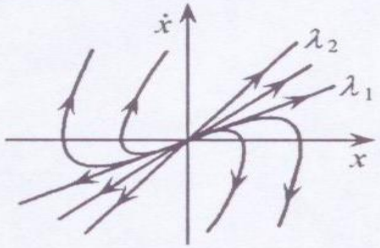
According to the position of $s_{1,2}$ in s-plane, there are six types of the singularity points:

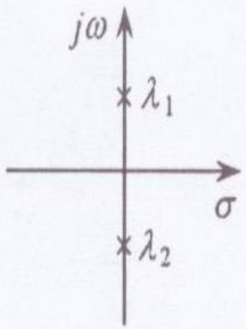
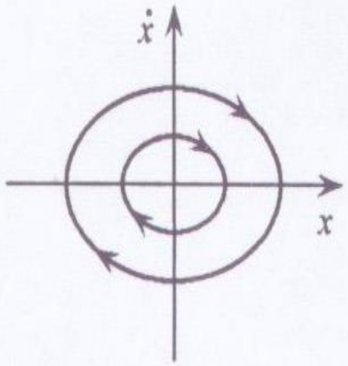
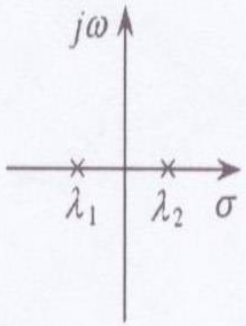
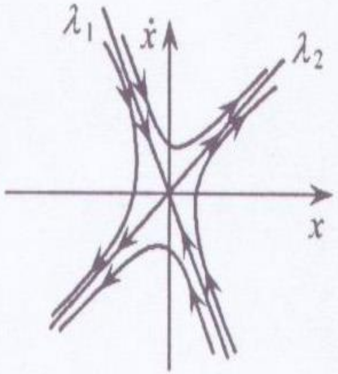
For $0 < \xi < 1 \Rightarrow$ stable focus; $\xi > 1 \Rightarrow$ stable nodes;

$-1 < \xi < 0 \Rightarrow$ unstable focus. $\xi < -1 \Rightarrow$ unstable nodes.

$\xi = 0 \Rightarrow$ center.

If : $\ddot{x} + 2\xi\omega_n\dot{x} - \omega_n^2 x = 0$ and $\xi > 0 \Rightarrow$ saddle point

types	Roots	Phase plane
stable focus		
unstable focus		
stable nodes		
unstable nodes		

centers		
saddle points		

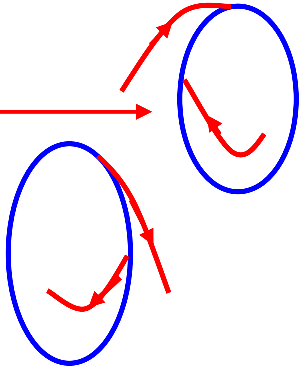
7.3.3 Analysis of the phase plane

2. limit cycle

A kind of phase locus with the closed loop form

→Corresponding to the self-oscillation.

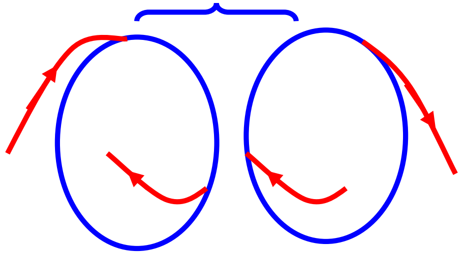
Types of the limit cycle :



(1) stable limit cycle

(2) unstable limit cycle

(3) semi - stable limit cycle



7.3.3 Analysis of the phase plane

Example 7.3.4:

The differential equation of the nonlinear contol system :

$$\ddot{x} + 0.5\dot{x} + 2x + x^2 = 0$$

Determine the singularity point of the system and plot the phase loci by the isocline method.

Solution make : $\dfrac{d\dot{x}}{dx} = \dfrac{-0.5\dot{x} - 2x - x^2}{\dot{x}} = \dfrac{0}{0}$

we have the singularity point : $x = 0,\ \dot{x} = 0;\ x = -2, \dot{x} = 0.$

Linearize the nonlinear differential equation to determine the types of the singularity points:

According to : $\ddot{x} = -f(x, \dot{x}) = -(0.5\dot{x} + 2x + x^2)$

we have : $\left.\dfrac{\partial f(x,\dot{x})}{\partial x}\right|_{x=0,\,\dot{x}=0} = 2 + 2x\big|_{x=0,\,\dot{x}=0} = 2$

7.3.3 Analysis of the phase plane

$$\left.\frac{\partial f(x,\dot{x})}{\partial \dot{x}}\right|_{x=0,\ \dot{x}=0} = 0.5$$

In the neighborhood of the singularity point (0, 0) the linearization equation of the system is:

$$\ddot{x} + 0.5\dot{x} + 2x = 0$$

The characteristic roots are: $s_{1,\ 2} = -0.25 \pm j1.39.$

So the singularity point (0, 0) is a stable focus.

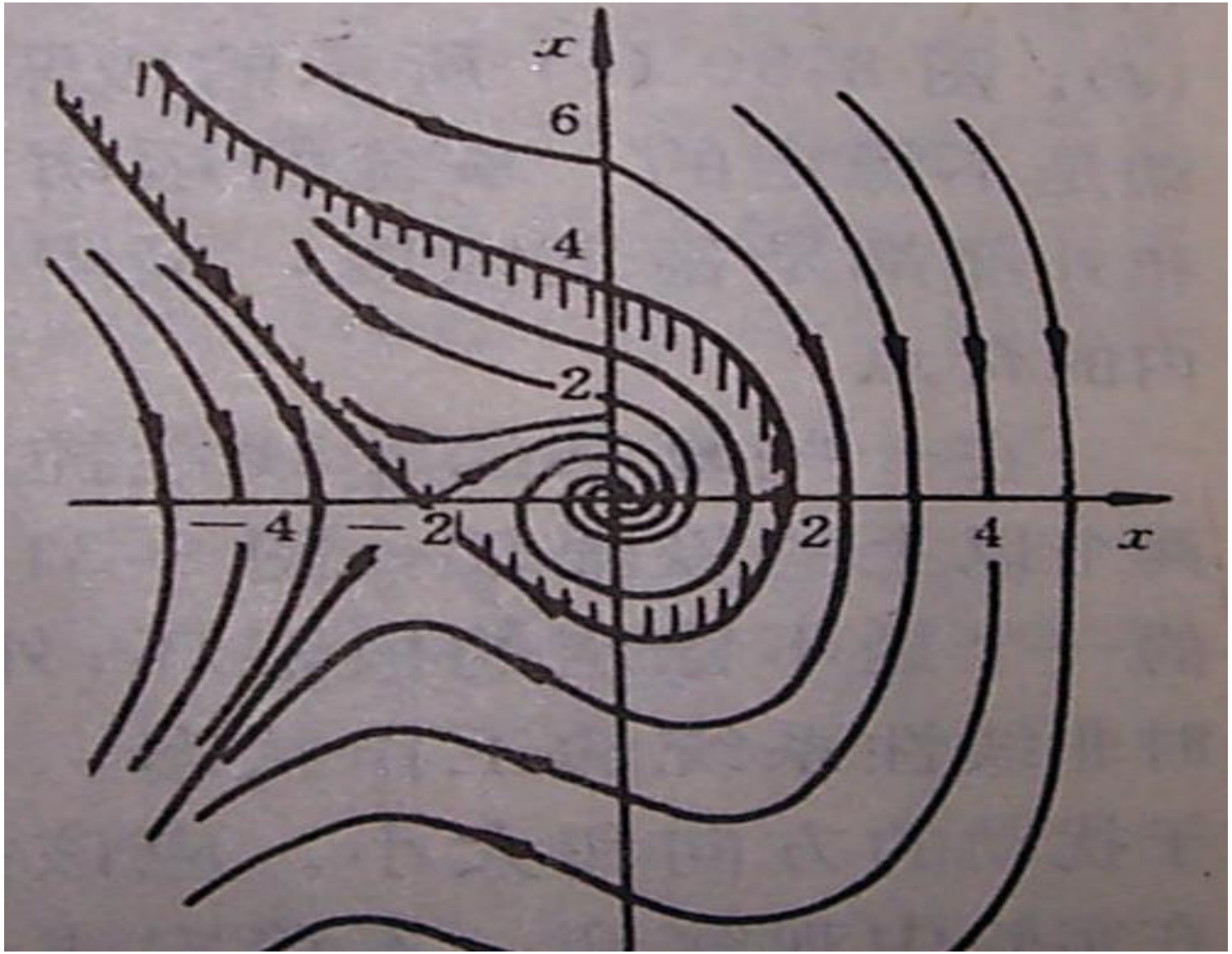
In the neighborhood of the singularity point (-2, 0) the linearization equation of the system is:

$$\ddot{x} + 0.5\dot{x} - 2x = 0$$

The characteristic roots are: $s_1 = 1.19,\ s_2 = -1.69.$

So the singularity point (-2, 0) is a saddle point.

The phase loci plotted by the isoclinal method are shown in following figure:



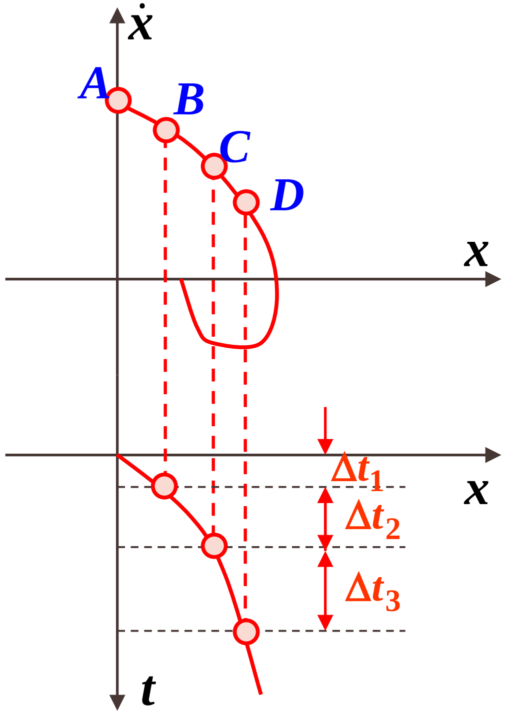
$\dot{x}$
6
4
2
-4
-2
2
4
x

7.3.3 Analysis of the phase plane

3. How to get the time response *x*(*t*) from the phase locus

Because : $\dot{x} = \frac{dx}{dt} \Rightarrow dt = \frac{dx}{\dot{x}} \Leftrightarrow \Delta t_i \approx \frac{\Delta x}{\dot{x}_{avi}}$

The graphical expression:



$$\Delta t_1 \approx \frac{\Delta x_1}{\dot{x}_{av1}} \Leftarrow \Delta x = x_B - x_A \,,\; \dot{x}_{av1} = \frac{\dot{x}_A + \dot{x}_B}{2}$$

$$\Delta t_2 \approx \frac{\Delta x_2}{\dot{x}_{av2}} \Leftarrow \Delta x_2 = x_C - x_B \,,\; \dot{x}_{av2} = \frac{\dot{x}_B + \dot{x}_C}{2}$$

$$\Delta t_3 \approx \frac{\Delta x_3}{\dot{x}_{av3}} \Leftarrow \Delta x_3 = x_D - x_C \,,\; \dot{x}_{av3} = \frac{\dot{x}_C + \dot{x}_D}{2}$$

We can get the time response curve *x*(*t*) from the phase locus to analyze the time specifications, such as the rise time t_r, Settling time t_s etc. , of the nonlinear systems.

7.3.3 Analysis of the phase plane

4. How to analyze the performance of the nonlinear systems from the phase locus

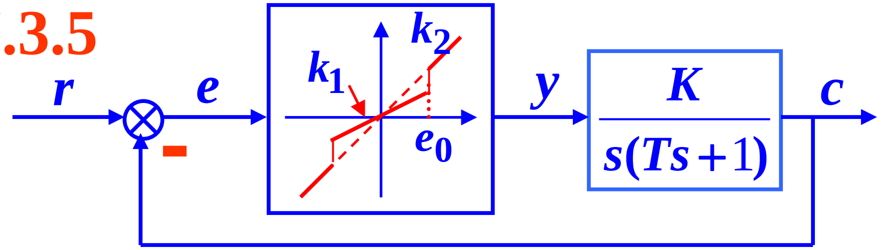
(1) We can analyze the stability directly from the phase locus: the phase locus is convergent or divergent.

(2) We can analyze the self-oscillation directly from the phase locus: the phase loci converge upon a limit circle.

(3) We can transform the phase locus into the time response curve $x(t)$ to analyze the rise time t_r , settling time t_s etc..

(4) Also we can analyze the steady state error, overshoot etc., directly from the phase locus.

Example 7.3.5



$k_1 = 0.0625$; $k_2 = 1$;

$T = 1$; $K = 4$; $e_0 = 0.2$

If $c(0) = \dot{c}(0) = 0$, analyze the unity step response of the system

7.3.3 Analysis of the phase plane

solution

$$because: \quad Y(s)\cdot\frac{4}{s(s+1)} = C(s) \Rightarrow y = \begin{cases} 0.0625e & e < 0.2 \\ e & e > 0.2 \end{cases}$$

$$We\ have: \quad \begin{cases} \ddot{e} + \dot{e} + 4e = 0 & |e| > 0.2 \\ \ddot{e} + \dot{e} + 0.25e = 0 & |e| < 0.2 \end{cases}$$

$$e(0) = 1, \ \dot{e}(0) = 0$$

Singularity points: (0, 0), and a stable nodes in the $e - \dot{e}$ plane.

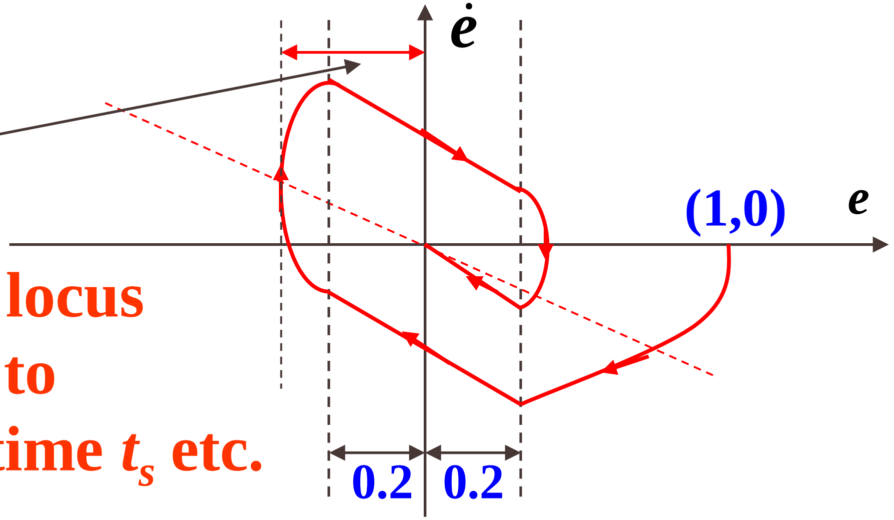
The phase locus is shown in following figure:

(1) Stability: stable

(2) Steady state error $e_{ss} = 0$

(3) Overshoot

(4) We can transform the phase locus into the time response curve $e(t)$ to analyze the rise time t_r, settling time t_s etc.



7.3 Phase plane method

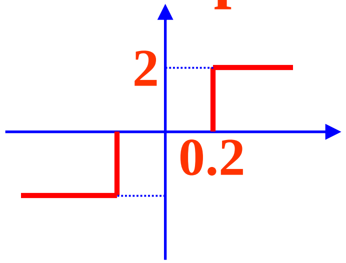
7.3.4 Attentions and development

(1) Phase plane method is only used for analyzing or designing the 1th-order or 2th-order nonlinear systems.

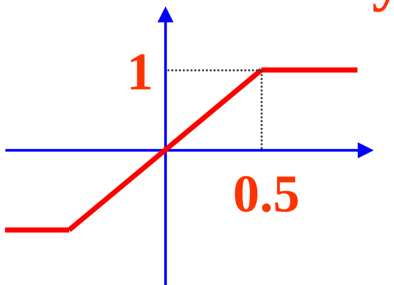
(2) Analyzing the nonlinear systems by phase plane method is more all-sided compare with the describing function method. but more complicated.

(3) Also the phase plane method is used to analyze the stability of some intelligent control systems, such as the Fuzzy control systems.

Exercise: For example 7.3.5, if the nonlinearity is:



(1)



(2)

Chapter 8 Discrete (Sampling) System

Chapter 8 Discrete (Sampling) System

8.1 Introduction

8.1.1 Sampling

Make a analog signal to be a discrete signal shown as in Fig.8.1 .

$x(t)$ —analog signal .

$x^*(t)$ —discrete signal .

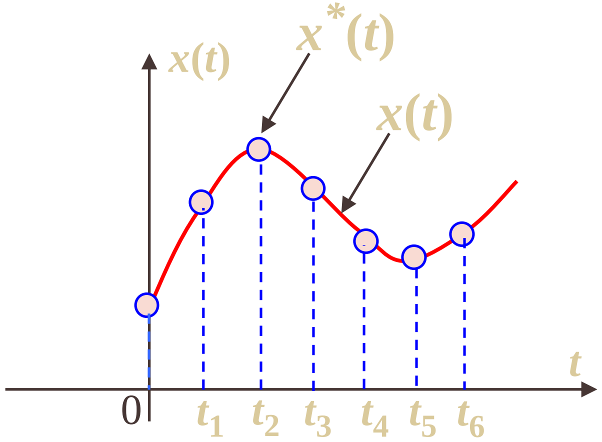


Fig.8.1 signal sampling

8.1.2 Ideal sampling switch —sampler

Sampler —the device which fulfill the sampling.

Another name —the sampling switch — which works like a switch shown as in Fig.8.2 .

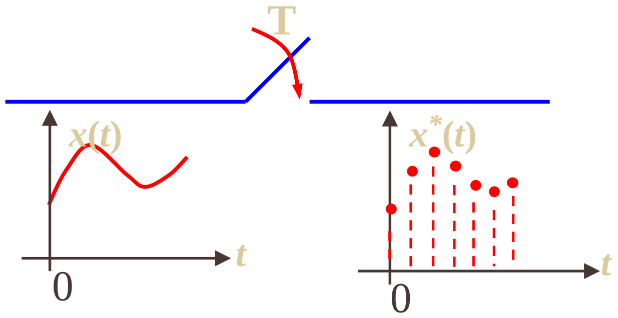


Fig.8.2 sampling switch

8.1.3 Some terms

1. Sampling period T— the time interval of the signal sampling: $T = t_{i+1} - t_i$.

Terimakasih