THEOREM 18.6

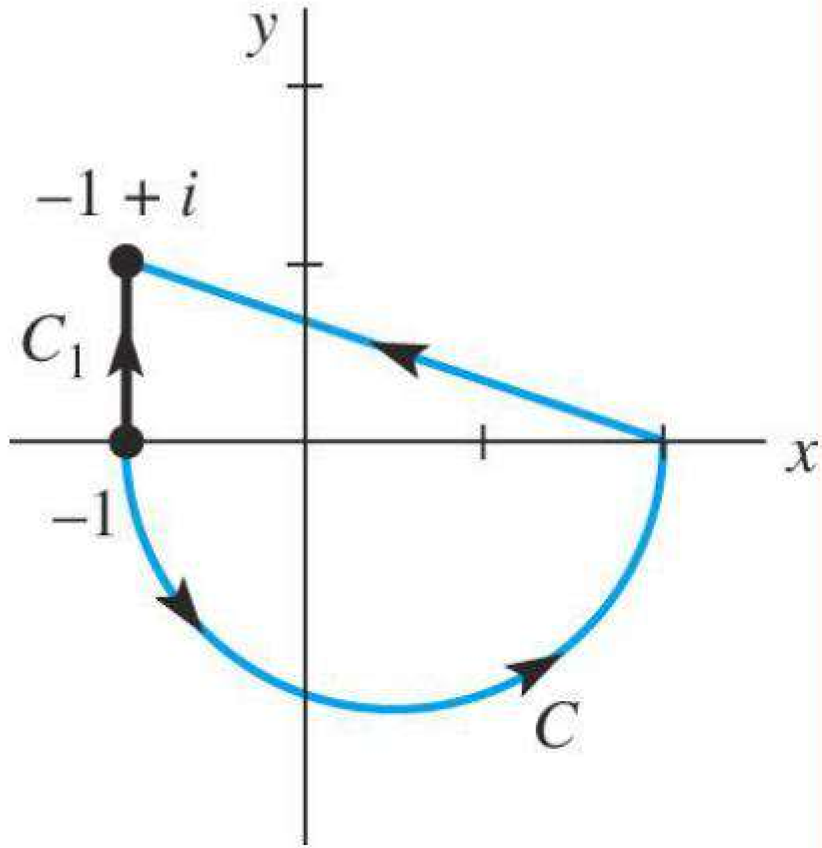
Analyticity Implies Path Independence

If f is an analytic function in a simply connected domain D, then $\int_C f(z)\,dz$ is independent of the path C.

Example 1

Evaluate $\int_C 2z\, dz$

where C is shown in Fig 18.20.



Example 1 (2)

Solution

Since $f(z) = 2z$ is entire, we choose the path C_1 to replace C (see Fig 18.20). C_1 is a straight line segment $x = -1$, $0 \le y \le 1$. Thus $z = -1 + iy$, $dz = idy$.

$$\int_C 2z\,dz = \int_{C_1} 2z\,dz$$

$$= -2\int_0^1 y\,dy - 2i\int_0^1 dy = -1 - 2i$$

DEFINITION 18.3

Antiderivative

Suppose f is continuous in a domain D. If there exists a function F such that $F'(z) = f(z)$ for each z in D, then F is called an **antiderivative** of f.

THEOREM 18.7

Fundamentals Theorem for Contour Integrals

Suppose f is continuous in a domain D and F is an antiderivative of f in D. Then for any contour C in D with initial point z_0 and terminal point z_1,

$$\int_C f(z)\,dz = F(z_1) - F(z_0) \tag{4}$$

THEOREM 18.7

Proof

With $F'(z) = f(z)$ for each z in D, we have

$$\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt = \int_a^b F'(z(t))z'(t)dt$$

$$= \int_a^b \frac{d}{dt} F(z(t))dt \qquad \leftarrow \text{Chain Rule}$$

$$= F(z(t))\Big|_a^b$$

$$= F(z(b)) - F(z(a)) = F(z_1) - F(z_0)$$

Example 2

In Example 1, the contour is from −1 to $-1 + i$. The function $f(z) = 2z$ is entire and $F(z) = z^2$ such that $F'(z) = 2z = f(z)$. Thus

$$\int_{-1}^{-1+i} 2z\,dz = z^2 \Big|_{-1}^{-1+i} = -1 - 2i$$

Example 3

Evaluate $\int_C \cos z dz$

where C is any contour fro $z = 0$ to $z = 2 + i$.

Solution

$$\int_C \cos z dz = \int_0^{2+i} \cos z dz = \sin z \Big|_0^{2+i}$$

$$= \sin(2+i) = 1.4031 - 0.4891i$$

Some Conclusions from Theorem 18.7

- If C is closed then $z_0 = z_2$, then

$$\oint_C f(z)\, dz = 0 \qquad (5)$$

- In other words:

  If a continuous function f has an antiderivative F in D, then $\int_C f(z)\, dz$ *is independent of the path.* (6)

- Sufficient condition for the existence of an antiderivative:

  If f is continuous and $\int_C f(z)\, dz$ *is independent of the path in a domain D, then f has an antiderivative everywhere in D.* (7)

THEOREM 18.8

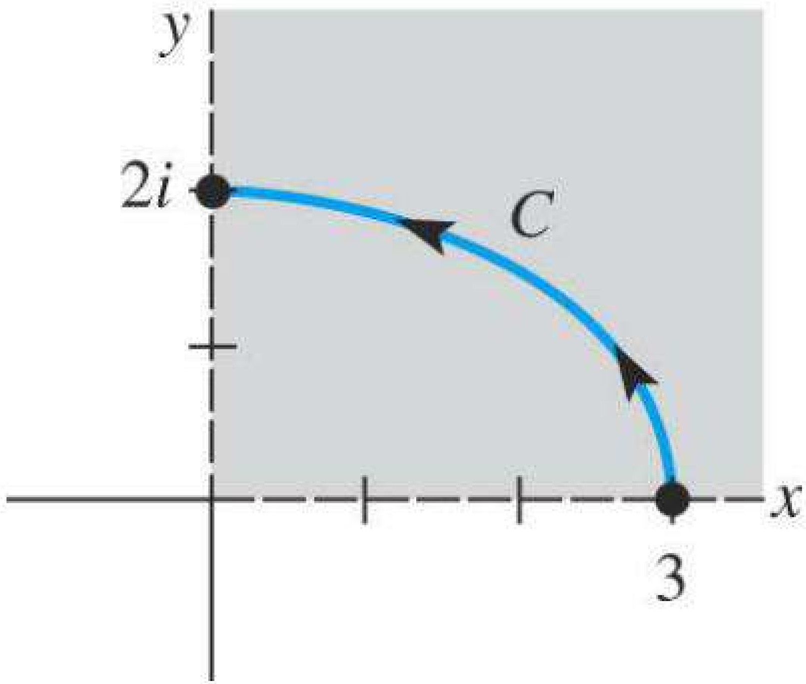
Existence of a Antiderivative

If f is analytic in a simply connected domain D, then f has an antiderivative in D; that is, there existence a function F such that $F'(z) = f(z)$ for all z in D.

Example 4

Evaluate $\int_C \frac{dz}{z}$

where C is shown in Fig 18.22.



Example 4 (2)

Solution

Suppose that D is the simply connected domain defined by $x > 0$, $y > 0$. In this case Ln z is an antiderivative of $1/z$. Hence

$$\int_3^{2i} \frac{dz}{z} = \text{Ln } z \Big|_3^{2i} = \text{Ln } 2i - \text{Ln } 3$$

$$\text{Ln } 2i = \log_e 2 + \frac{\pi}{2} i, \ \text{Ln } 3 = \log_e 3$$

$$\int_3^{2i} \frac{dz}{z} = \log_e \frac{2}{3} + \frac{\pi}{2} i$$

18.4 Cauchy Integral Formulas

THEOREM 18.9 **Cauchy's Integral Formula**

Let f be analytic in a simply connected domain D, and let C be a simple closed contour lying entirely within D. If z_0 is any point within C, then

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0}\,dz \qquad (1)$$

THEOREM 18.9

Proof

Let C_1 be a circle centered at z_0 with radius small enough that it is interior to C. Then we have

$$\oint_C \frac{f(z)}{z - z_0}\, dz = \oint_{C_1} \frac{f(z)}{z - z_0}\, dz \qquad (2)$$

For the right side of (2)

$$\oint_{C_1} \frac{f(z)}{z - z_0}\, dz = \oint_{C_1} \frac{f(z_0) - f(z_0) + f(z)}{z - z_0}\, dz \qquad (3)$$

$$= f(z_0) \oint_{C_1} \frac{dz}{z - z_0} + \oint_{C_1} \frac{f(z) - f(z_0)}{z - z_0}\, dz$$

THEOREM 18.9 proof

From (4) of Sec. 18.2, we know

$$\oint_C \frac{dz}{z - z_0} = 2\pi i$$

Thus (3) becomes

$$\oint_{C_1} \frac{f(z)}{z - z_0}\, dz = 2\pi i f(z_0) + \oint_{C_1} \frac{f(z) - f(z_0)}{z - z_0}\, dz \qquad (4)$$

However from the ML-inequality and the fact that the length of C_1 is small enough, the second term of the right side in (4) is zero. We complete the proof.

$$\left| \oint_{C_1} \frac{f(z) - f(z_0)}{z - z_0}\, dz \right| \le \frac{\delta}{\delta/2} 2\pi \left(\frac{\delta}{2} \right) = 2\pi\varepsilon$$

- A more practical restatement of Theorem 18.9 is :

If f is analytic at all points within and on a simple closed contour C, and z_0 is any point interior to C, then

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0}\, dz \qquad (5)$$

Example 1

Evaluate $\oint_C \frac{z^2 - 4z + 4}{z + i}\, dz$

where C is the circle $|z| = 2$.

Solution

First $f = z^2 - 4z + 4$ is analytic and $z_0 = -i$ is within C. Thus

$$\oint_C \frac{z^2 - 4z + 4}{z + i}\, dz = 2\pi i f(-i) = 2\pi i(3 + 4i) = 2\pi(-4 + 3i)$$

Example 2

Evaluate $\oint_C \frac{z}{z^2+9}\,dz$

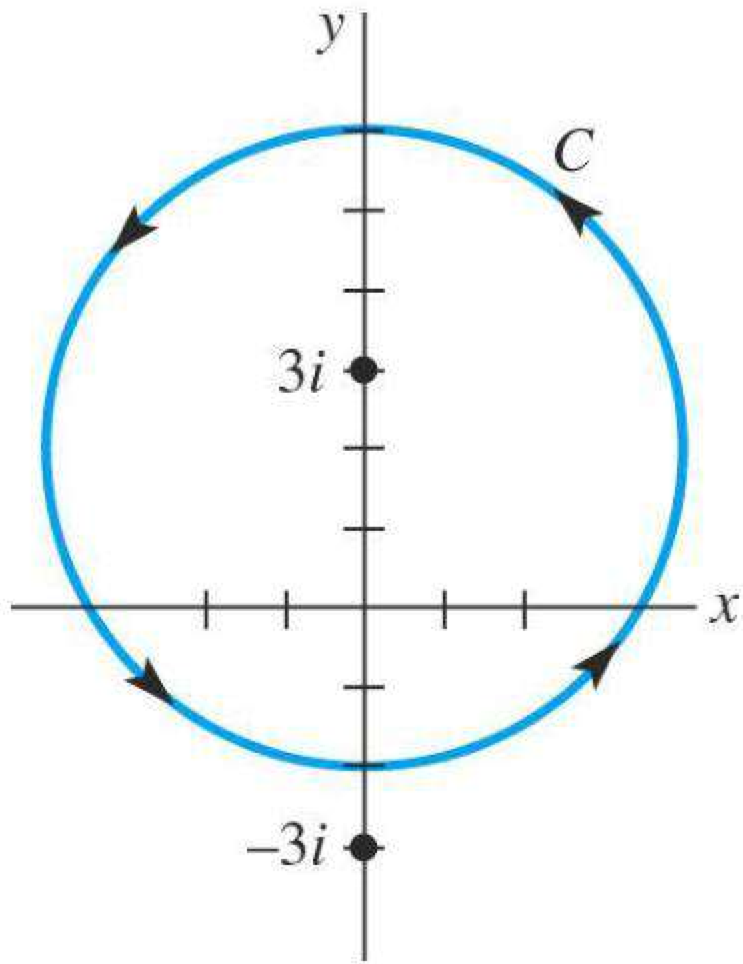
where C is the circle $|z - 2i| = 4$.

Solution

See Fig 18.25. Only $z = 3i$ is within C, and

$$\frac{z}{z^2+9} = \frac{\frac{z}{z+3i}}{z-3i}$$

Fig 18.25



Example 2 (2)

Let $f(z) = \dfrac{z}{z+3i}$, then

$$\oint_C \frac{z}{z^2+9}\,dz = \oint_C \frac{\dfrac{z}{z+3i}}{z-3i}\,dz = 2\pi i f(3i) = 2\pi i \frac{3i}{6i} = \pi i$$

Example 3

The complex function $f(z) = k/(\bar{z} - \bar{z}_1)$ where $k = a + ib$ and z_1 are complex numbers, gives rise to a flow in the domain $z \neq z_1$. If C is a simple closed contour containing $z = z_1$ in its interior, then we have

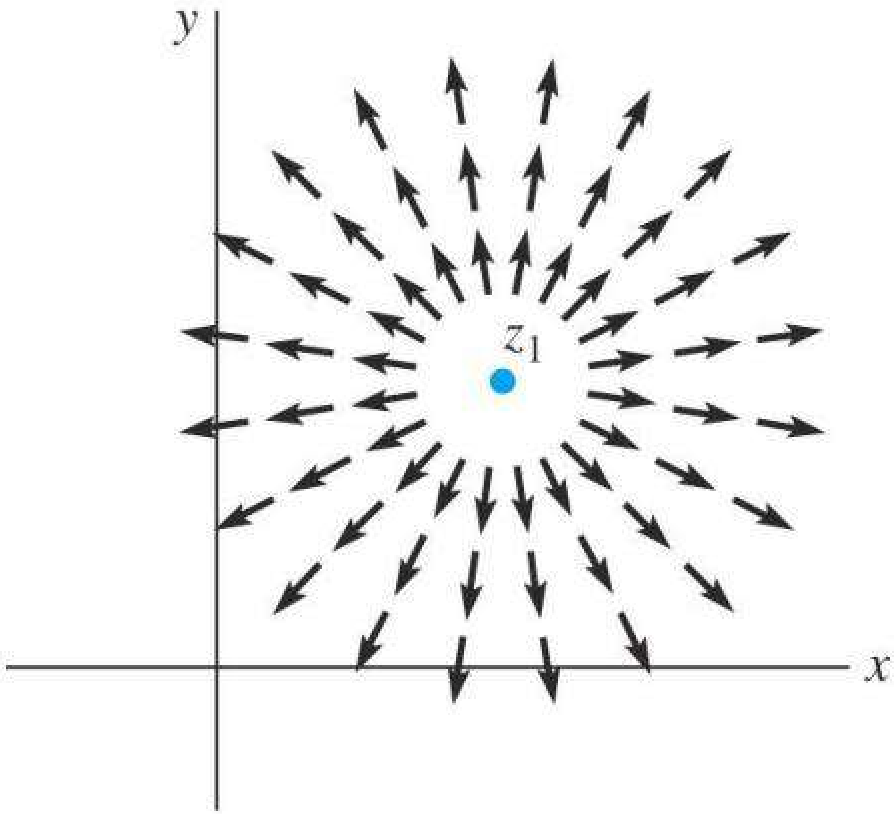
$$\oint_C \overline{f(z)}\, dz = \oint_C \frac{a - ib}{z - z_1}\, dz = 2\pi i(a - ib)$$

Example 3 (2)

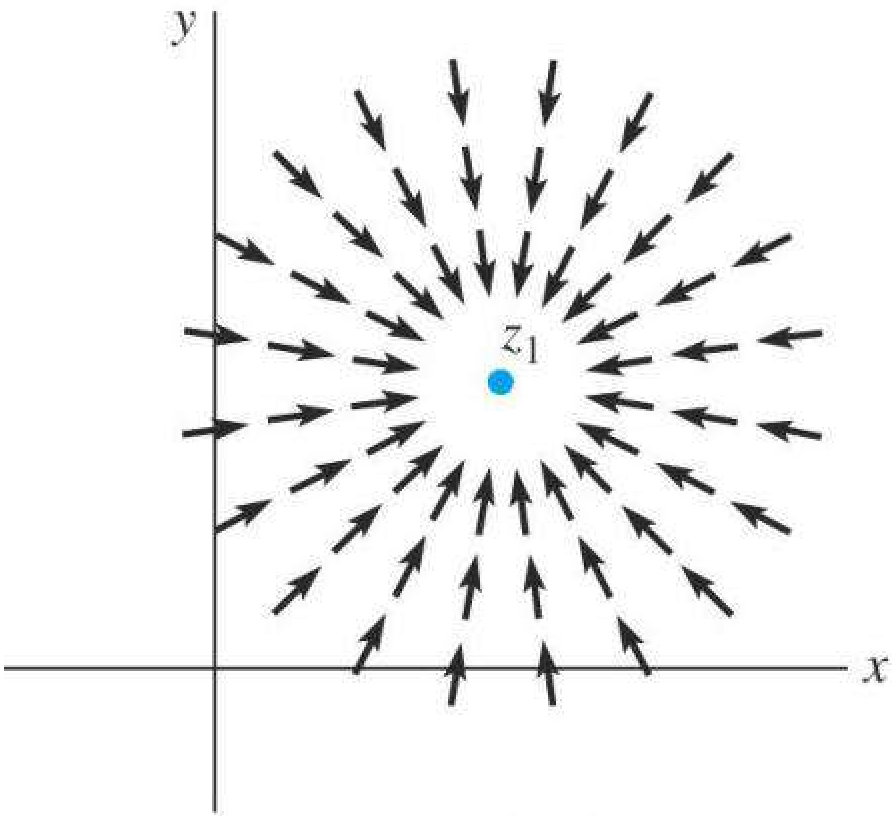
The circulation around C is $2\pi b$ and the net flux across C is $2\pi a$. If z_1 were in the exterior of C both of them would be zero. Note that when k is real, the circulation around C is zero but the net flux across C is $2\pi k$. The complex number z_1 is called a source when $k > 0$ and is a sink when $k < 0$.

See Fig 18.26.

Fig 18.26



(a) Source: $k > 0$



(b) Sink: $k < 0$

THEOREM 18.10

Cauchy's Integral Formula For Derivative

Let f be analytic in a simply connected domain D, and let C be a simple closed contour lying entirely within D. If z_0 is any point within C, then

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z - z_0)^{n+1}}\,dz \tag{6}$$

THEOREM 18.10

Partial Proof

Prove only for $n = 1$. From the definition of the derivative and (1): $f(z) = k/(\bar{z} - \bar{z}_1)$

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{1}{2\pi i \Delta z} \left[\oint_C \frac{f(z)}{z - (z_0 + \Delta z)} dz - \oint_C \frac{f(z)}{z - z_0} dz \right]$$

$$= \lim_{\Delta z \to 0} \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0 - \Delta z)(z - z_0)} dz$$

THEOREM 18.10 Partial Proof

From the ML-inequality and

$$\left| \oint_C \frac{f(z)}{(z-z_0)^2}\,dz - \oint_C \frac{f(z)}{(z-z_0-\Delta z)(z-z_0)}\,dz \right|$$

$$= \left| \oint_C \frac{-\Delta z f(z)}{(z-z_0)^2(z-z_0-\Delta z)}\,dz \right| \le \frac{2ML|\Delta z|}{\delta^3} \to 0 \text{ as } \Delta z \to 0$$

Thus

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^2}\,dz$$

Example 4

Evaluate $\oint_C \frac{z+1}{z^4+4z^3}\,dz$

where C is the circle $|z| = 1$.

Solution

This integrand is not analytic at $z = 0, -4$ but only $z = 0$ lies within C. Since

$$\frac{z+1}{z^4+4z^3} = \frac{\frac{z+1}{z+4}}{z^3}$$

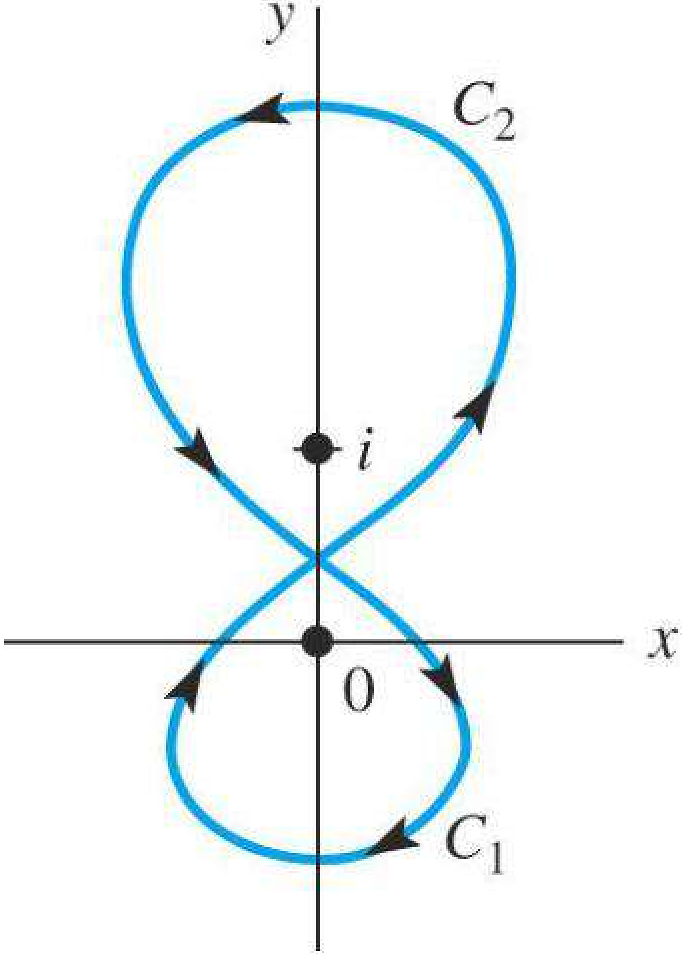
We get $z_0 = 0$, $n = 2$, $f(z) = (z + 1)/(z + 4)$, $f''(z) = -6/(z + 4)^3$. By (6):

$$\oint_C \frac{z+1}{z^4+4z^3}\,dz = \frac{2\pi i}{2!} f''(0) = -\frac{3\pi}{32} i$$

Example 5

Evaluate $\oint_C \frac{z^3+3}{z(z-i)^2}\,dz$

where C is shown in Fig 18.27.



Example 5 (2)

Solution

Though C is not simple, we can think of it is as the union of two simple closed contours C_1 and C_2 in Fig 18.27.

$$\oint_C \frac{z^3+3}{z(z-i)^2}\,dz = \oint_{C_1} \frac{z^3+3}{z(z-i)^2}\,dz + \oint_{C_2} \frac{z+3}{z(z-i)^2}\,dz$$

$$= -\oint_{C_1} \frac{\dfrac{z^3+3}{z(z-i)^2}}{z}\,dz + \oint_{C_2} \frac{\dfrac{z^3+3}{z}}{z(z-i)^2}\,dz$$

$$= -I_1 + I_2$$

Example 5 (3)

For I_1 : $z_0 = 0$, $f(z) = (z^3 + 3)/(z - i)^2$:

$$I_1 = \oint_{C_1} \frac{\frac{z^3+3}{z(z-i)^2}}{z}\, dz = 2\pi i f(0) = -6\pi i$$

For I_2 : $z_0 = i$, $n = 1$, $f(z) = (z^3 + 3)/z$, $f\,'(z) = (2z^3 - 3)/z^2$:

$$I_2 = \oint_{C_2} \frac{\frac{z^3+3}{z}}{(z-i)^2}\, dz = \frac{2\pi i}{1!} f'(i) = -2\pi i(3+2i) = 2\pi(-2+3i)$$

We get

$$\oint_C \frac{z^3+3}{z(z-i)^2}\, dz = -I_1 + I_2 = 6\pi i + 2\pi(-2+3i) = 4\pi(-1+3i)$$

Liouville's Theorem

- If we take the contour C to be the circle $|z - z_0| = r$, from (6) and ML-inequality that

$$|f^{(n)}(z_0)| = \frac{n!}{2\pi}\left|\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz\right|$$

$$\leq \frac{n!}{2\pi} M \frac{1}{r^{n+1}} 2\pi r = \frac{n!M}{r^n} \qquad (7)$$

where $|f(z)| \leq M$ for all points on C. The result in (7) is called Cauchy's inequality.

THEOREM 18.11 **Liouville's Theorem**

The only bounded entire functions are constants.

Proof

For $n = 1$, (7) gives $|f'(z_0)| \le M/r$. By taking r arbitrarily large, we can make $|f'(z_0)|$ as small as we wish. That is, $|f'(z_0)| = 0$, f is a constant function.