# PROSES ACAK

Kuliah Statistik Dan Stokastik
Teknik Fisika ITS

# Konsep Proses Acak

- Proses acak / stokastik X(t) didefinisikan:
  - Variabel acak yg berubah thd waktu: nilainya berubah thd waktu
  - Fungsi waktu yg acak: kumpulan fungsi yg ditentukan utk setiap outcome
- X(t) pada $t_1$ adalah variabel acak $X(t_1)$
- Realisasi X(t) adalah sebuah fungsi waktu disebut sample function/path
- Ruang sampel-nya (dikenal sbg ensemble) adalah kumpulan fungsi waktu
- X(t) pd waktu yg berbeda dapat memiliki distribusi yg sangat berbeda

# Contoh 1

Sebuah sinyal sinus $X(t) = \cos(t+\phi)$ dengan $\phi$ adalah RV yg ditentukan oleh pelempaan koin:

$\phi = 0$ jika muncul muka

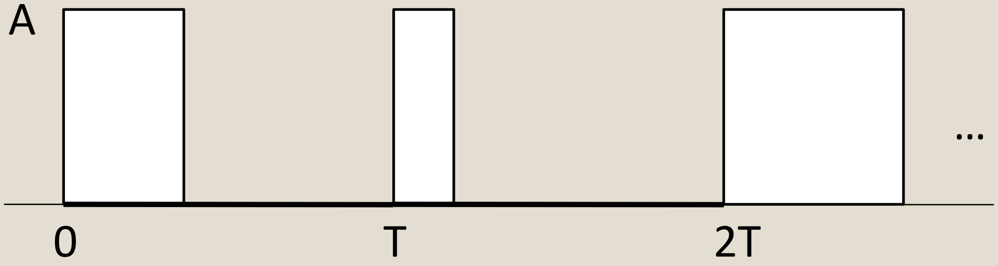
$\phi = 1$ jika muncul belakang

- X(t) adalah acak yg nilainya tdk dpt ditentukan sebelumnya
- X(t) bukan RV krn sebuah fungsi waktu bukan angka yg ditentukan oleh outcome eksperimen acak
- Ruang sampel = $\{x_1(t), x_2(t)\} = \{\cos(t), -\cos(t)\}$ krn $\cos(t+\pi) = -\cos(t)$
- Utk sembarang waktu $t_1$, $X(t_1)$ adalah RV dua nilai yg paling mungkin: $\cos(t_1)$ dan $-\cos(t_1)$

# Contoh 2

Sebuah sinyal pulsa kotak dengan periode T, tinggi pulsa A, dan lebar pulsa W yang bebas dr periode ke peiode dan terdistribusi uniform sepanjang satu periode: W ~ U(0,T)

- X(t) adalah acak krn nilainya tdk dpt ditentukan secara pasti sebelumnya
- X(t) bukan RV krn nilainya adalah fungsi waktu meskipun utk sembarang $t_1$, $X(t_1)$ adalah RV biner dng dua nilai yg mungkin: 0 atau A
- Ruang sampel adalah kumpulan fungsi waktu yg tak terbatas.



# Karakterisasi Proses Acak (1)

- Karena $X(t_1)$ adalah RV, maka:
  - fungsi mean =

$$\bar{x}(t_1) = E[X(t_1)] = \int_{-\infty}^{\infty} x f_{X(t_1)}(x)\, dx$$

  - fungsi mean square =

$$E[X^2(t_1)] = \int_{-\infty}^{\infty} x^2 f_{X(t_1)}(x)\, dx$$

  - fungsi variansi =

$$\sigma^2_{X(t1)} = E[X^2(t_1)] - [x(t_1)]^2$$

# Karakterisasi Proses Acak (2)

- Terdapat joint PDF $f_{X(t1),X(t2)}(x_1,x_2)$ utk dua RV $X(t_1)$ dan $X(t_2)$, shg:
  - fungsi autokorelasi =

$$R_x(t_1,t_2) = E[X(t_1)X(t_2)]$$

  - fungsi autokovarian =

$$C_x(t_1,t_2) = \mathrm{E}[\{X(t_1) - \bar{x}(t_1)\}\{X(t_2) - \bar{x}(t_2)\}]$$

  - koefisien korelasi =

$$\rho(t_1,t_2) = \frac{C_x(t_1,t_2)}{\sqrt{C_x(t_1,t_1)C_x(t_2,t_2)}} = \frac{C_x(t_1,t_2)}{\sigma_{x(t_1)}\sigma_{x(t_2)}}$$

Autokovarian = autokorelasi – (mean $t_1$)(mean $t_2$)

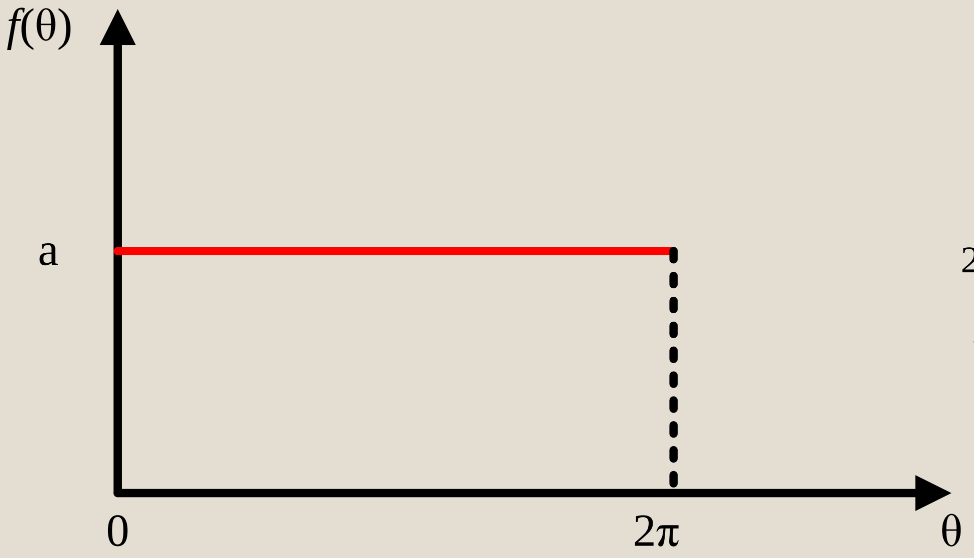
$$C_x(t_1,t_2) = R_x(t_1,t_2) - \bar{x}(t_1)\bar{x}(t_2)$$

# Contoh 3

Tentukan nilai mean dan fungsi autokorelasi dari:

$$X(t) = A\cos(\omega t + \theta)$$

jika A dan ω konstan, sedangkan θ berdistribusi uniform pada interval (0,2π).




$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

$$\int_0^{2\pi} a\,d\theta = 1 \rightarrow a.2\pi = 1$$

$$a = \frac{1}{2\pi}$$

Fungsi kerapatan probabilitas dari θ adalah: $f_\theta(\theta)=\frac{1}{2\pi}$

$$
\begin{aligned}
E[X(t)] &= E[g(\theta)] \\
&= \int_{-\infty}^{\infty} g(\theta) f_\theta(\theta) d\theta \\
&= \int_{0}^{2\pi} A\cos(\omega t+\theta)\frac{1}{2\pi} d\theta \\
&= \frac{A}{2\pi}\int_{0}^{2\pi} \cos(\omega t+\theta) d\theta \\
&= \frac{A}{2\pi}\sin(\omega t+\theta)\Big|_{0}^{2\pi} \\
&= \frac{A}{2\pi}\left\{\sin(\omega t+2\pi)-\sin(\omega t+0)\right\} \\
&= \frac{A}{2\pi}\left\{\sin\omega t\cos 2\pi+\cos\omega t\sin 2\pi-\sin\omega t\right\} \\
&= \frac{A}{2\pi}\left\{\sin\omega t-\sin\omega t\right\} \\
&= 0
\end{aligned}
$$

$$R_{XX}(t_1,t_2)=E\left[X(t_1)X(t_2)\right]$$

$$=E\left[\left(A\cos(\omega t_1+\theta)\right)\left(A\cos(\omega t_2+\theta)\right)\right]$$

$$=E\left[A^2\cos(\omega t_1+\theta)\cos(\omega t_2+\theta)\right]$$

$$=A^2E\left[\cos(\omega t_1+\theta)\cos(\omega t_2+\theta)\right]$$

$$=\frac{A^2}{2}E\left[\cos(\omega t_1+\theta+\omega t_2+\theta)+\cos(\omega t_1+\theta-\omega t_2-\theta)\right]$$

$$=\frac{A^2}{2}E\left[\cos\left(\omega(t_1+t_2)+2\theta\right)+\cos\omega(t_1-t_2)\right]$$

$$=\frac{A^2}{2}E\left[\cos\left(\omega(t_1+t_2)+2\theta\right)\right]+\frac{A^2}{2}E\left[\cos\omega(t_1-t_2)\right]$$

$$E\left[\cos\left(\omega\left(t_1+t_2\right)+2\theta\right)\right]=\int_0^{2\pi}\cos\left(\omega\left(t_1+t_2\right)+2\theta\right)\frac{1}{2\pi}d\theta$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\cos\left(\omega\left(t_1+t_2\right)+2\theta\right)d\theta$$

$$=\frac{1}{2\pi}\frac{1}{2}\sin\left(\omega\left(t_1+t_2\right)+2\theta\right)\Big|_0^{2\pi}$$

$$=\frac{1}{4\pi}\left\{\sin\left(\omega\left(t_1+t_2\right)+4\pi\right)-\sin\omega\left(t_1+t_2\right)\right\}$$

$$=\frac{1}{4\pi}\left\{\sin\omega\left(t_1+t_2\right)\cos 4\pi+\cos\omega\left(t_1+t_2\right)\sin 4\pi-\sin\omega\left(t_1+t_2\right)\right\}$$

$$=\frac{1}{4\pi}\left\{\sin\omega\left(t_1+t_2\right)-\sin\omega\left(t_1+t_2\right)\right\}$$

$$=0$$

$$R_{xx}\left(t_1,t_2\right)=\frac{A^2}{2}E\left[\cos\omega\left(t_2-t_1\right)\right]$$

$$=\frac{A^2}{2}\cos\omega\left(t_2-t_1\right)$$

$$=R_{xx}\left(t_2-t_1\right)$$

# Hubungan Definisi dalam Ensemble dan Waktu

$$\bar{x} = E\left[x(t)\right] = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x(t)dt$$

$$\overline{x^2} = E[x(t)^2] = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x(t)^2 dt$$

$$R_{xx}(\tau) = E\left[x(t)x(t+\tau)\right] = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x(t)x(t+\tau)dt$$

$$R_{xy}(\tau) = E\left[x(t)y(t+\tau)\right] = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x(t)y(t+\tau)dt$$

# Sifat Fungsi Autokorelasi

1. $R_{XX}(\tau)$ fungsi genap, sehingga : $R_{XX}(\tau)=R_{XX}(-\tau)$
   Bukti : $R_{XX}(\tau)=E\left[X\left(t_1\right)X\left(t_2\right)\right]$
   $=E\left[X\left(t_2\right)X\left(t_1\right)\right]$
   $=E\left[X\left(t+\tau\right)X\left(t\right)\right]$
   $=R_{XX}\left(t-\left(t+\tau\right)\right)$
   $=R_{XX}(-\tau)$

2. Apabila $\left|R_{XX}(\tau)\right|$ tdk punya komponen periodik, maka

$$\lim_{\tau\to\infty} R_{XX}(\tau)=\overline{X}^2$$

3. $R_{XX}(0)=E\left[X^2(t)\right]$
   Bukti:

$$E\left[X^2(t)\right]=E\left[X(t)X(t)\right]$$
$$=R_{XX}(t-t)$$
$$=R_{XX}(0)$$

4. $\left|R_{XX}(\tau)\right|\leq R_{XX}(0)$
5. Bila x(t) periodik, maka $R_{XX}(\tau)$ juga periodik dengan periode yang sama

# Sifat Fungsi Korelasi Silang

1. $R_{XY}(\tau)$ fungsi ganjil

   Bukti :

   $$R_{XY}(\tau) = E\left[X(t)Y(t+\tau)\right]$$
   $$= E\left[Y(t+\tau)X(t)\right]$$
   $$= R_{YX}\left(t-(t+\tau)\right)$$
   $$= R_{YX}(-\tau)$$

   $$R_{XY}(\tau) \neq R_{XY}(-\tau)$$

   Jadi bukan fungsi genap

2. $\left|R_{XY}(\tau)\right| \leq \sqrt{R_{XX}(0)R_{YY}(0)}$

3. Jika :

   $$\dot{X}(t) \triangleq \frac{d}{dt}X(t)$$

   maka:

$$R_{\dot{x}y}(\tau) \triangleq E\left[\dot{X}(t+\tau)Yt)\right] = \frac{d}{d\tau}R_{xy}(\tau)$$

$$R_{\dot{x}\dot{y}}(\tau) \triangleq E\left[\dot{X}(t+\tau)\dot{Y}t)\right] = -\frac{d^2}{d\tau^2}R_{xy}(\tau)$$