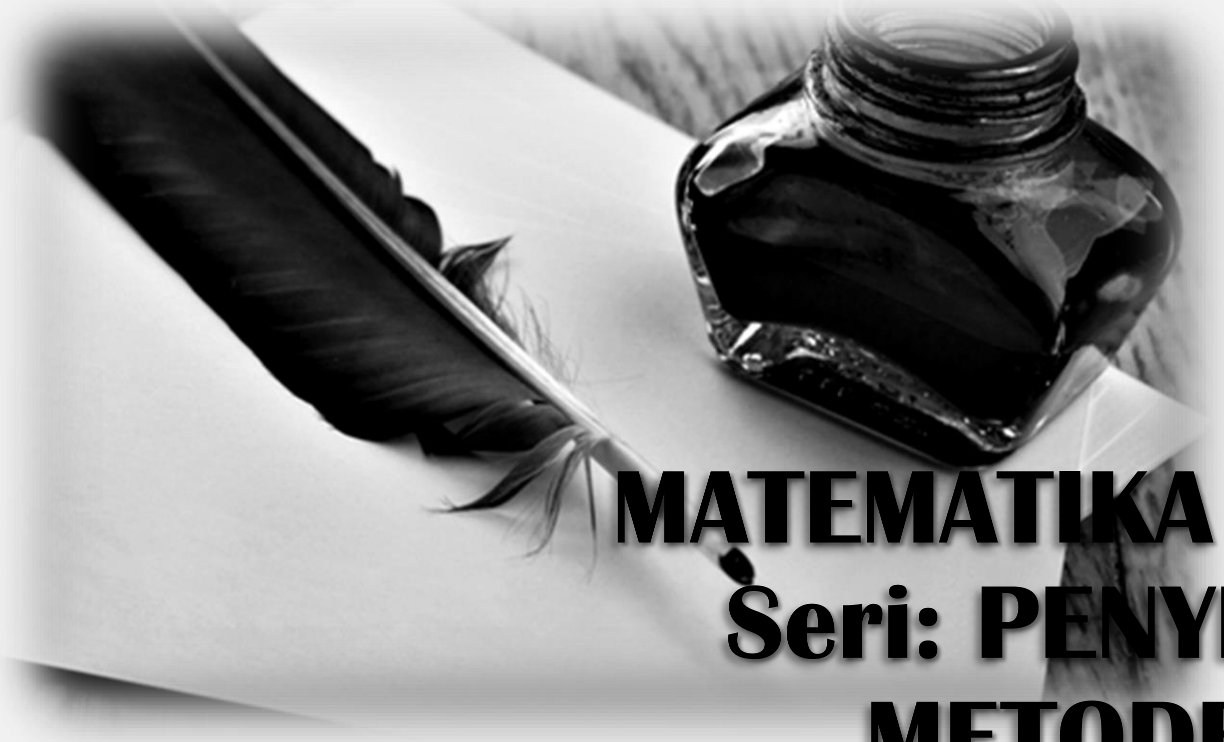
Institut Teknologi Sepuluh Nopember - Surabaya



MATEMATIKA REKAYASA II
Seri: PENYELESAIAN PD METODE FROBENIUS

Desain: Aulia Siti Aisjah

Klasifikasi Persamaan Diferensial

Deferensial

- Biasa (Xi, i=1)
  - Turunan Pertama
    - Linear (derajat=1) y’+p(x)y=r(x)
      - Homogeneous r(x)=0
      - Non Homogeneous r(x) 0
    - Non Linear (derajat>1) $y'+p(x)y=g(x)y^a$
  - Turunan Kedua y’’+p(x)y’+q(x)y=0 p,q (constant)
  - Turunan ke-n
- Parsial (Xi, i>1)

Persamaan Diferensial Linear Koefisien tidak konstan

$$y''+\frac{c(t)}{m}y'+\frac{k(t)}{m}y=0$$

Pengantar
Materi
Contoh Soal
Ringkasan
Latihan
Asesmen

Latihan :

$$y = \sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + \cdots$$

$$y' = \sum_{m=1}^{\infty} m a_m x^{m-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

$$y'' = \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} = 2a_2 + 3.2 a_3 x + 4.3 a_4 x^2 + \cdots$$

. . .

Extended Power Series (Deret Pangkat yang dikembangkan)

$$y(x) = \sum_{m=0}^{\infty} a_m (x - x_0)^m$$

Tentukan y’ dan y”

Bentuk extended power series digunakan untuk menyelesaikan PD?

Pengantar
Materi
Contoh Soal
Ringkasan
Latihan
Asesmen

Frobenius Method

Kelebihan metoda ini adalah aplikasinya yang lebih umum, dimana Metoda Deret Pangkat tidak bisa lakukan

Metode Frobenius

- **TEOREMA 1:**

Untuk semua jenis persamaan differensial yang memenuhi persamaan:

$$y'' + \frac{b(x)}{x} y' + \frac{c(x)}{x^2} y = 0 \qquad (1)$$

mempunyai sekurang-kurangnya satu solusi yang dapat diwakili oleh:

$$y(x) = x^r \sum_{m=0}^{\infty} a_m x^m = x^r (a_0 + a_1 x + a_2 x^2 + ...) \qquad (2)$$

dimana pangkat adalah bilangan riil atau kompleks yang dipilih sehingga

$$a_0 \neq 0$$

Gunakan deret pangkat yang dikembangkan untuk meny. PD

Perhatikan hasil peny. Pada koefisien x pangkat terendah → pers indicial

Untuk persamaan indicial dengan 2 akar

Kasus 1 untuk 2 akar berbeda, yaitu r1 dan r2 dan keduanya bernilai real

$$y_1(x) = x^{r_1}(a_0 + a_1 x + a_2 x^2 + \cdots)$$

$$y_2(x) = x^{r_2}(A_0 + A_1 x + A_2 x^2 + \cdots)$$

Kasus 2: r1 = r2 = r

$$\boxed{y_1(x) = x^r(a_0 + a_1 x + a_2 x^2 + \cdots) \qquad [r = \tfrac{1}{2}(1 - b_0)]}$$

dan

$$\boxed{y_2(x) = y_1(x)\ln x + x^r(A_1 x + A_2 x^2 + \cdots) \qquad (x > 0).}$$

Kasus 3, dua akar r1 dan r2 berbeda dan selisihnya adalah bilangan integer

r1 – r2 = bilangan integer

$$\boxed{y_1(x) = x^{r_1}(a_0 + a_1 x + a_2 x^2 + \cdots)}$$

$$\boxed{y_2(x) = k y_1(x) \ln x + x^{r_2}(A_0 + A_1 x + A_2 x^2 + \cdots),}$$

Cotoh Soal

Bentuk PD berikut → apakah mempunyai kesamaan dengan bentuk PD khusus dari Frobenius?

$$9x^2y'' + 3x^2y' + 2y = 0$$

$$\begin{aligned} 9x^2y'' + 3x^2y' + 2y &= 0 \\ 9x^2\left(\sum_{n=0}^{\infty} c_n x^{n+r}\right)'' + 3x^2\left(\sum_{n=0}^{\infty} c_n x^{n+r}\right)' + 2\left(\sum_{n=0}^{\infty} c_n x^{n+r}\right) &= 0 \end{aligned}$$

$$9x^2y'' + 3x^2y' + 2y = 0$$

$$\begin{aligned} 9x^2y'' + 3x^2y' + 2y &= 0 \\ 9x^2\sum_{n=0}^{\infty} c_n(n+r)(n+r-1)x^{n+r-2} + 3x^2\sum_{n=0}^{\infty} c_n(n+r)x^{n+r-1} + 2\sum_{n=0}^{\infty} c_n x^{n+r} &= 0 \end{aligned}$$

$$9x^2y'' + 3x^2y' + 2y = 0$$

$$\begin{aligned}
9x^2y'' + 3x^2y' + 2y &= 0 \\
9x^2 \sum_{n=0}^{\infty} c_n(n+r)(n+r-1)x^{n+r-2} + 3x^2 \sum_{n=0}^{\infty} c_n(n+r)x^{n+r-1} + 2\sum_{n=0}^{\infty} c_n x^{n+r} &= 0 \\
\sum_{n=0}^{\infty} 9(n+r)(n+r-1)c_n x^{n+r} + \sum_{n=0}^{\infty} 3(n+r)c_n x^{n+r+1} + \sum_{n=0}^{\infty} 2c_n x^{n+r} &= 0 \\
\sum_{k=0}^{\infty} 9(k+r)(k+r-1)c_k x^{k+r} + \sum_{k=1}^{\infty} 3(k+r-1)c_{k-1} x^{k+r} + \sum_{k=0}^{\infty} 2c_k x^{k+r} &= 0 \\
(9r(r-1)+2)c_0 x^r + \sum_{k=1}^{\infty} \left[9(k+r)(k+r-1)c_k + 3(k+r-1)c_{k-1} + 2c_k\right] x^{k+r} &= 0
\end{aligned}$$

$$9x^2y'' + 3x^2y' + 2y = 0$$

Perhatikan untuk koefisien dari x pangkat terendah

Pers. indicial

$$\begin{aligned}
(9r(r-1)+2)c_0 &= 0 \\
9r(r-1)+2 &= 0 \\
9r^2 - 9r + 2 &= 0 \\
r_{1,2} &= \frac{9 \pm \sqrt{81-72}}{18} \\
&= \frac{9 \pm 3}{18} \\
&= \frac{2}{3}, \frac{1}{3}
\end{aligned}$$

- Contoh (A) :
- Selesaikan ***4xy”+2y’ +y= 0***

$$y' = \sum_{m=0}^{\infty}(m+r)a_m x^{m+r-1} \qquad y'' = \sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r-2}$$

$$4x\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r-2} + 2\sum_{m=0}^{\infty}(m+r)a_m x^{m+r-1} + \sum_{m=0}^{\infty}a_m x^{m+r} = 0$$

Pers. Indicial

4r(r–1)+2r=0 → r(2r–1) =0 → r =0 or *r=1/2 (Kasus 1)*

$$4\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r-1} + 2\sum_{m=0}^{\infty}(m+r)a_m x^{m+r-1} + \sum_{m=0}^{\infty}a_m x^{m+r} = 0$$

(saat r=0)

$$4\sum_{m=0}^{\infty} m(m-1)a_m x^{m-1} + 2\sum_{m=0}^{\infty} m a_m x^{m-1} + \sum_{m=0}^{\infty} a_m x^m = 0$$

$$4\sum_{s=0}^{\infty} (s+1)s a_{s+1} x^s + 2\sum_{s=0}^{\infty} (s+1) a_{s+1} x^s + \sum_{s=0}^{\infty} a_s x^s = 0$$

$[4(s+1)s+2(s+1)]a_{s+1} + a_s=0$ → recurrence $\quad a_{s+1} = -\dfrac{1}{(4s+2)(s+1)} a_s$

$\therefore\ a_1= -a_0/2!\ ;\ \ a_2= -a_1/12=a_0/4!\ ;\ \ a_3= -a_2/30= -a_0/6!\ \ \ldots$

→ $a_0=1$

$$y_1 = x^0\left(a_0 - \frac{a_0}{2!}x + \frac{a_0}{4!}x^2 - \frac{a_0}{6!}x^3 + \ldots\right) = 1 - \frac{1}{2!}x + \frac{1}{4!}x^2 - \frac{1}{6!}x^3 + \ldots = \cos\sqrt{x}$$

(saat r=1/2)

$$4\sum_{m=0}^{\infty}(m+\frac{1}{2})(m-\frac{1}{2})a_m x^{m-1/2} + 2\sum_{m=0}^{\infty}(m+\frac{1}{2})a_m x^{m-1/2} + \sum_{m=0}^{\infty}a_m x^{m+1/2} = 0$$

$$4\sum_{m=0}^{\infty}(s+1)sa_{s+1/2}x^s + 2\sum_{m=0}^{\infty}(s+1)a_{s+1/2}x^s + \sum_{m=0}^{\infty}a_{s-1/2}x^s = 0$$

$[4(s+1)s+2(s+1)]a_{s+1} + a_s=0$ → rekurensi $a_{s+1/2} = -\frac{1}{(4s+2)(s+1)}a_{s-1/2}$

$\therefore\ a_1= -a_0/3!$ (s=1/2); $a_2= -a_1/20=a_0/5!$ (s=3/2); $a_3= -a_2/42=a_0/7!$ (s=5/2); …

$a_0=1$

$\therefore$

$$y_2 = x^{1/2}(a_0 - \frac{a_0}{3!}x + \frac{a_0}{5!}x^2 - \frac{a_0}{7!}x^3 + \ldots) = x^{1/2} - \frac{1}{3!}x^{3/2} + \frac{1}{5!}x^{5/2} - \frac{1}{7!}x^{7/2} + \ldots = \sin\sqrt{x}$$

Teorema ke 2

Metode Frobenius – untuk Kasus 2
Dari tiga kasus

Kasus 2 – akar dobel sama $r_1 = r_2 = r$.

Penyelesaian

$$y_1(x) = x^r(a_0 + a_1x + a_2x^2 + \cdots) \qquad \left[r = \tfrac{1}{2}(1 - b_0)\right]$$

$$y_2(x) = y_1(x)\ln x + x^r(A_1x + A_2x^2 + \cdots) \qquad (x > 0).$$

Contoh (B) : Selesaikan:

$(x^2-x)y'' + (3x-1)y' + y = 0$

Dengan metode Frobenius

$$y' = \sum_{m=0}^{\infty}(m+r)a_m x^{m+r-1} \qquad y'' = \sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r-2}$$

$$(x^2-x)\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r-2} + (3x-1)\sum_{m=0}^{\infty}(m+r)a_m x^{m+r-1} + \sum_{m=0}^{\infty}a_m x^{m+r} = 0$$

$$\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r} - \sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r-1} + 3\sum_{m=0}^{\infty}(m+r)a_m x^{m+r} -$$

$$\sum_{m=0}^{\infty}(m+r)a_m x^{m+r-1} + \sum_{m=0}^{\infty}a_m x^{m+r} = 0$$

$-r(r-1)-r=0 \rightarrow r^2=0 \rightarrow r = 0$ (kasus 2.)

(saat r=0)

$\therefore\ -(s+1)^2\, a_{s+1} + (s+1)^2 a_s = 0 \rightarrow$ recurrence $a_{s+1} = a_s$

$\therefore\ a_0 = a_1 = a_2 = a_3 = a_4 = \ldots$

$a_0 = 1$

$$\sum_{m=0}^{\infty} m(m-1)a_m x^m - \sum_{m=0}^{\infty} m(m-1)a_m x^{m-1} + 3\sum_{m=0}^{\infty} m a_m x^m - \sum_{m=0}^{\infty} m a_m x^{m-1} + \sum_{m=0}^{\infty} a_m x^m = 0$$

$$\sum_{m=0}^{\infty} s(s-1)a_s x^s - \sum_{m=0}^{\infty} (s+1)s a_{s+1} x^s + 3\sum_{m=0}^{\infty} s a_s x^s - \sum_{m=0}^{\infty} (s+1)a_{s+1} x^s + \sum_{m=0}^{\infty} a_s x^s = 0$$

$$y_1 = (a_0 + a_0 x + a_0 x^2 + a_0 x^3 + \ldots) = 1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}$$

THEOREM 2

Metode Frobenius. Kasus ke 3

Case 3. akar berbeda - integer.

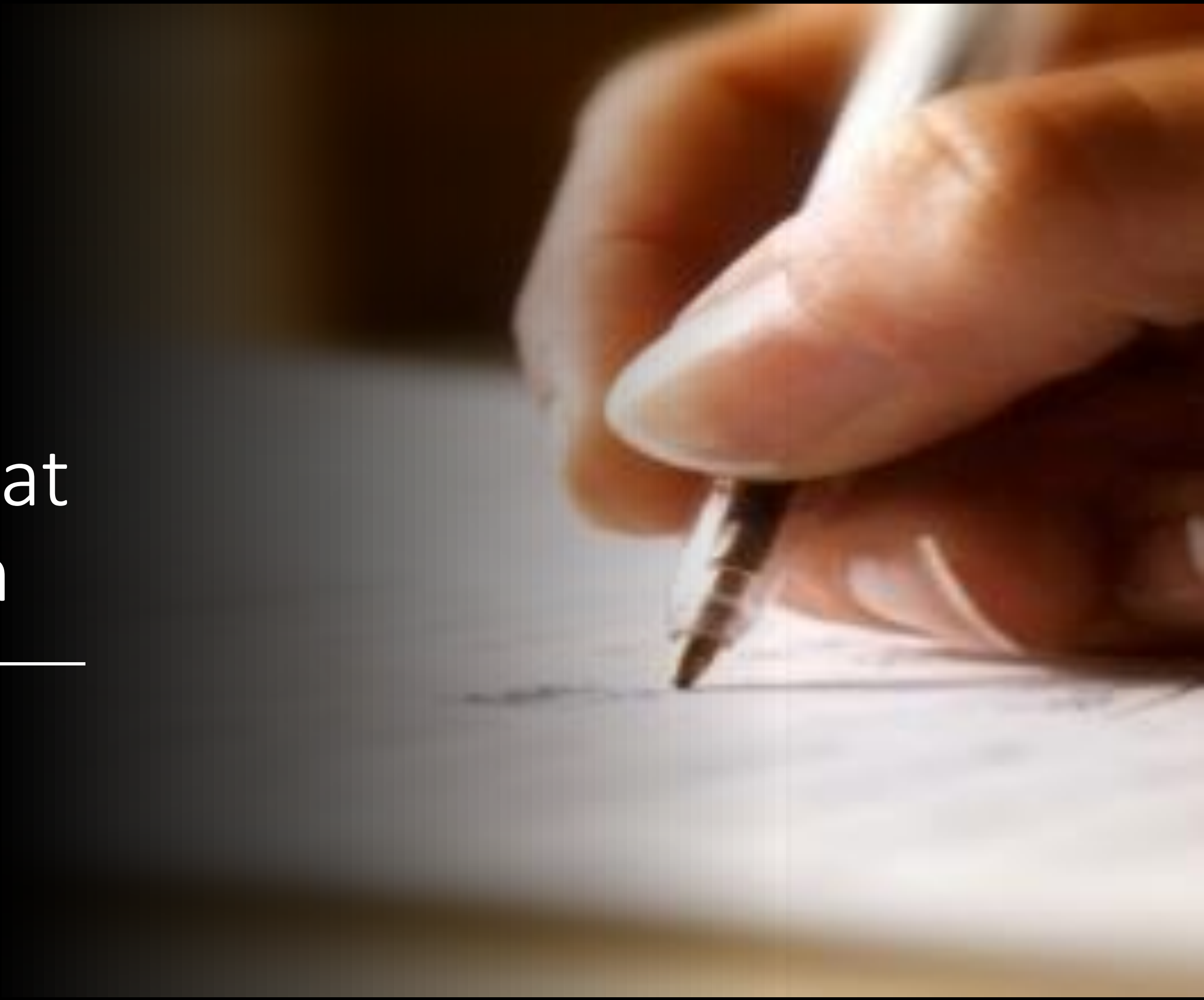
(9) $$y_1(x) = x^{r_1}(a_0 + a_1 x + a_2 x^2 + \cdots)$$

(sama dengan bentuk sebelumnya)

(10) $$y_2(x) = k y_1(x) \ln x + x^{r_2}(A_0 + A_1 x + A_2 x^2 + \cdots),$$

Akar – akar $r_1 – r_2 > 0$ dan k tidak sama 0

Catat semua penjelasan saat kuliah sinkron



Terimakasih